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EXISTENCE AND CONCENTRATION OF SOLUTIONS FOR ELLIPTIC EQUATIONS INVOLVING THE FRACTIONAL P-LAPLACIAN OPERATOR

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Introduction

1. Reasons for choosing the topic

Multi-phase partial differential equations are widely utilized to characterize direction-dependent phenomena, with a prominent application in image processing, specifically in image denoising via diffusion-based heat models.

In the isotropic case, the linear diffusion mechanism smooths the image uniformly in all directions, thereby leading to the blurring of edges and boundaries. Conversely, multi-phase anisotropic diffusion models enable the description of directional selective diffusion, which smooths homogeneous regions to mitigate noise while preserving edges and boundaries. Models such as the Perona–Malik equation serve as classic examples.

Beyond denoising applications, multi-phase equations are also employed in image inpainting, allowing structural information to propagate along directional features such as edges, contours, or textures, thus preserving the original geometry of the image.

In physics and materials science, these equations play a crucial role in the design of anisotropic media, where physical properties exhibit directional dependence.

In tandem with local equations, numerous modern models implement nonlocal operators to model long-range interactions. Operators such as the fractional Laplacian $(-\Delta)^s$ and the fractional p-Laplacian $(-\Delta)_p^s$ naturally arise in a plethora of problems, including optimization, mathematical finance, anomalous diffusion, quantum mechanics, materials science, fluid dynamics, and mathematical biology.

Studies on equations involving nonlocal operators generally focus on three main directions:

1. Investigating the existence of weak solutions to fractional partial differential equations.
2. Examining qualitative properties of solutions, such as regularity, symmetry, and asymptotic behavior.
3. Analyzing the blow-up of solutions to diffusion-type equations (such as heat and wave propagation equations).

In this thesis, we dedicate our focus to the first research direction. Within this avenue, the topics of investigation are remarkably diverse due to the emergence of various types of equations. Kirchhoff-type problems originate from the equation of the form

$$\rho u_{tt} - \Big(\frac{p_0}{h} + \frac{E}{2L}\int_0^L |u_x|^2 dx\Big) u_{xx} = 0, \tag{1}$$

where ρ, L, E, h, p_0 are constants possessing specific physical meanings. Kirchhoff introduced this model as an extension of D'Alembert's classical wave equation to describe the transverse vibrations of a stretched string.

The Choquard equation is given by

$$-\Delta v + V(x)v = (I_\alpha * |v|^p)|v|^{p-2}v \quad \text{in } \mathbb{R}^N,\ p > 2,$$

where I_α is the Riesz potential defined for each $x \in \mathbb{R}^N \setminus \{0\}$. P. Choquard was the first to study the Choquard equation to describe an electron oscillating around its orbital region. This equation also arises in the theory of polarons at rest, as well as in models describing the interaction between non-relativistic quantum mechanics and gravity. In 2019, Teng and Agarwal investigated the existence of non-negative ground state solutions and the non-existence of ground state solutions for the fractional Choquard-Schrödinger-Poisson equation as follows

$$(-\Delta)^s v + V(x)v + K(x)\phi_v^t |v|^{q-2}v = (I_\alpha * |v|^p)|v|^{p-2}v \quad \text{in } \mathbb{R}^3. \tag{2}$$

In 2020, Ambrosio and Radulescu published a recent work on the existence of weak solutions for an equation involving the double-phase (p,q)-Laplacian operator of the form:

$$(-\Delta)_p^s v + (-\Delta)_q^s v + V(\zeta x)(|v|^{p-2}v + |v|^{q-2}v) = \mathfrak{f}(v) \text{ in } \mathbb{R}^N, \tag{3}$$

where $\zeta > 0$ is a small parameter, $s \in (0,1), 1 < p < q < \dfrac{N}{s}$, $\mathfrak{f}$ is a continuous function with subcritical growth in the fractional Sobolev space, and V satisfies certain conditions. In 2022, Zhang, Tang, and Radulescu studied the double-phase fractional equation in the case of critical and supercritical growth in $\mathbb{R}^N$ of the form

$$(-\Delta)_p^s v + (-\Delta)_q^s v + V(\zeta x)(|v|^{p-2}v + |v|^{q-2}v) = h(v) + |v|^{r-2}v, \tag{4}$$

where $r = q_s^*$ or $r > q_s^*$, and $h : \mathbb{R} \to \mathbb{R}$ is a continuous function. Note that as $s \to 1^-$, equation (3) reduces to

$$-\Delta_p v - \Delta_q v + V(\zeta x)(|v|^{p-2}v + |v|^{q-2}v) = \mathfrak{f}(v) \text{ in } \mathbb{R}^N. \tag{5}$$

Another direction related to the Born–Infeld equation, which is based on a variational modification of Maxwell's Lagrangian density and arises in electromagnetism, electrostatics, and electrodynamics, serves as the inspiration for the following anisotropic equation in $\mathbb{R}^N$:

$$-\text{div}\left(\frac{\nabla u}{(1-2|\nabla u|^2)^{1/2}}\right) = h(u).$$

Indeed, by using the Taylor expansion, we obtain

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(2n-1)!!}(n-1)!\Delta_{2n} u.$$

In 2024, Ambrosio, Zheng, and co-authors investigated the multiplicity and concentration of solutions for the following problem:

$$(-\Delta)_p^s v + (-\Delta)_q^s v + V(\zeta x)(|v|^{p-2}v + |v|^{q-2}v) = \left(\frac{1}{|x|^\mu} * F(v)\right)\mathfrak{f}(u). \tag{6}$$

Ambrosio proved the existence and multiplicity of solutions for (6) by employing the penalization method, Ljusternik–Schnirelmann category theory, and variational methods. Compared to the problem proposed in this thesis, the assumptions on V and $\mathfrak{f}$ are different. Furthermore, we do not utilize the penalization method due to the conditions imposed on the potential V. The equation

$$(-\Delta)_p^s v + V(\zeta x)|v|^{p-2}v = (\frac{1}{|x|^\mu} * F(v))\mathfrak{f}(v) \quad \text{in } \mathbb{R}^N, \tag{7}$$

was studied by Ambrosio in. In addition, for the case $p = 2$, in 2020, Chen, Li, and Yang investigated the multiplicity and concentration of solutions for the equation:

$$\zeta^{2s}(-\Delta)^s v + V(x)v = \zeta^{\mu-N}\Big(\frac{1}{|x|^\mu} * F(v)\Big)\mathfrak{f}(v) + |v|^{2_s^*-2}v \text{ in } \mathbb{R}^N, \tag{8}$$

where V satisfies condition (V) and $\mathfrak{f}$ satisfies appropriate assumptions.

A singular Schrödinger equation is given by

$$-div(|\nabla u|^{N-2}\nabla u) + V(x)|u|^{N-2}u = \frac{f(x,u)}{|x|^\eta}, x \in \mathbb{R}^N, \tag{9}$$

where $N \geq 2$, $0 \leq \eta < N$, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function, and $f(x,s)$ is continuous on $\mathbb{R}^N \times \mathbb{R}$. When $\eta = 0$, equation (9) reduces to the classical non-singular elliptic equation, which was first investigated by Cao in 1992 for the case $N = 2$.

In 2020, Thin initiated a research direction on the existence of solutions for singular equations driven by the fractional p-Laplacian operator in $\mathbb{R}^N$ of the form

$$\mathcal{L}_p^s u(x) + V(x)|u|^{p-2}u = \frac{f(x,u)}{|x|^\gamma}, \tag{10}$$

where $\gamma \in [0, N)$, $s \in (0,1)$, and $\mathcal{L}_p^s$ is the fractional p-Laplacian operator defined by

$$\mathcal{L}_p^s\varphi(x) = 2\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} |\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))K(x-y)dy,$$

with $x \in \mathbb{R}^N$, φ being a sufficiently smooth function, and $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ being a measurable function.

In 2022, Yanjun and Chungen investigated the existence of ground state solutions for a singular Kirchhoff-Schrödinger type equation with exponential growth nonlinearities in $\mathbb{R}^N$ as follows:

$$M\left(\int_{\mathbb{R}^N} |\nabla u|^N + V(x)\,|u|^N\,dx\right)\left(-\Delta_N u + V(x)\,|u|^{N-2}\,u\right) = \frac{f(x,u)}{|x|^\eta}, \tag{11}$$

where $N \geq 2$, $0 < \eta < N$, and $\Delta_N u = div(|\nabla u|^{N-2}\,\nabla u)$. Here, V is a continuous function bounded from below by a positive constant, and $f(x,t)$ exhibits critical exponential growth.

In 2023, Zhang, Han, and Thin employed the Mountain Pass Theorem to establish the existence of solutions for a Schrödinger-Kirchhoff type equation involving the fractional p-Laplacian operator of the form

$$\begin{aligned} M&\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy + \int_{\mathbb{R}^N} V(x)\,|u(x)|^p\,dx\right) \\ &\times\left(\mathcal{L}_p^s u(x) + V(x)\,|u|^{p-2}\,u\right) = K(x)f(x,u), \end{aligned} \tag{12}$$

where M is a continuous Kirchhoff function, $0 < s < 1 < p < \infty$, $sp = N$, f is a continuous function on $\mathbb{R}^N \times \mathbb{R}$ with critical exponential growth, and K, V are positive continuous functions satisfying appropriate assumptions.

Motivated by the aforementioned studies, we identify several **open problems** that warrant further investigation as follows:

Open Problem 1: In 2020, Chen, Li, and Yang investigated the existence of solutions for the Choquard equation (8) involving the fractional Laplacian and the critical exponent $|v|^{2_s^*-2}v$. Ambrosio studied the existence of weak solutions for the Choquard equation driven by the fractional p-Laplacian (7). Furthermore, Zheng and co-authors investigated the multiplicity and concentration of solutions for the Choquard equation involving the fractional (p,q)-Laplacian (6). Therefore, a natural problem arises to extend the results of Chen–Li–Yang and Ambrosio to multi-phase fractional Choquard equations with critical exponents.

Open Problem 2: Thin investigated the existence of weak solutions for the singular equation (10) involving the fractional p-Laplacian with exponential growth, while Yanjun and Chungen studied the existence of solutions for a singular Kirchhoff-type problem (11) driven by the p-Laplacian with exponential growth. A natural question is to extend Thin's results to Kirchhoff-type equations.

In this thesis, motivated by both the scientific significance and practical applications of classes of nonlocal operators, Kirchhoff-type problems, and multi-phase partial differential equations, we investigate the existence of solutions for multi-phase fractional Kirchhoff equations. We anticipate that the combination of these models will provide a more comprehensive description of numerous physical, biological, and chemical systems. We aim to address **Open Problems 1 and 2**. Specifically, this thesis investigates the existence of weak solutions for the following fractional equations:

1. Multi-phase fractional Choquard equations

$$\sum_{i=1}^{m}((-\Delta)^s_{p_i}u+\eta_1|u|^{p_i-2}u)=\eta_2^2[\frac{1}{|x|^\mu}*H_1(u)]h_1(u)+\eta_3h_2(u),$$

where the nonlinearities exhibit Trudinger–Moser type exponential growth. Furthermore, we also investigate the existence of solutions for multi-phase fractional Choquard equations with critical growth

$$\begin{aligned}&\sum_{i=1}^{m}(-\Delta)^s_{p_i}v+\gamma_1\sum_{i=1}^{m}|v|^{p_i-2}v\\&\qquad=\gamma_2^2\lambda\Big[\frac{1}{|x|^\mu}*F(v)\Big]\mathfrak{f}(v)+(v^+)^{q_s^*-2}v^+.\quad (P_{\gamma_1,\gamma_2})\end{aligned}$$

The existence of solutions to (P_{γ_1,γ_2}) serves as a foundational basis for proving the existence of solutions to the subsequent equation.

2. Multi-phase non-autonomous Choquard equations involving fractional operators

$$\begin{aligned}&\sum_{i=1}^{m}(-\Delta)^s_{p_i}v+\sum_{i=1}^{m}V(\zeta x)|v|^{p_i-2}v\\&=\lambda\left(\frac{1}{|x|^\mu}*Q(\zeta y)F(v)\right)Q(\zeta x)\mathfrak{f}(v)+(v^+)^{q_s^*-2}v^+,v^+(x)=\max\{v(x),0\}.\end{aligned}$$

3. Finally, we consider the singular multi-phase fractional Kirchhoff–

Schrödinger equation on $\mathbb{R}^N$ as follows:

$$\sum_{i=1}^{T} M_i\Big(\iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^{p_i} K_i(x-y)dxdy + \int_{\mathbb{R}^N} V(x)\,|u(x)|^{p_i}\,dx\Big)$$
$$\times\,(\mathcal{L}^s_{p_i}u(x) + V(x)\,|u|^{p_i-2}\,u) = \frac{f(x,u)}{|x|^{\gamma}}.$$

The choice of our topic "**Existence and concentration of solutions for elliptic equations involving the fractional p-Laplacian operator**" aims to enrich the existing results on the existence of weak solutions for partial differential equations driven by the fractional p-Laplacian.

2. Research objectives and objects of study

In this thesis, our objective is to establish the existence of weak solutions and the concentration phenomena of solutions for certain classes of multi-phase fractional partial differential equations.

Our objects of study are multi-phase fractional equations and fractional Sobolev spaces.

3. Thesis overview

a) Regarding the first research topic, we have:

- Proved the existence of weak solutions for equations (1.18) and (1.36). The main results of this chapter are Theorem 1.2.5 and Theorem 1.3.3. To establish these results, we first show that the energy functional corresponding to the problem satisfies the geometric structure of the Mountain Pass Theorem. This implies the existence of a (PS) sequence with an estimated mountain-pass level. Next, by employing Lions-type compactness criteria, we demonstrate that the weak limit of the (PS) sequence is a nontrivial weak solution of the given problem.

b) Regarding the second research topic:

- We establish the existence of weak solutions and the concentration behavior of solutions for equation (2.1). The main results of this chapter are Theorem 2.2.4 and Theorem 2.2.5. Theorem 2.2.4 demonstrates the existence of weak solutions for the non-autonomous problem (2.1) when ζ is sufficiently small, and shows that the global maximum of the solution concentrates around the global minimum of the potential V and the global maximum of the potential Q. Theorem 2.2.5 states that the sequence of solutions for problem (2.1), after a proper translation modification, converges to the solution of the limit problem. Such results are absent in [6], [9], [22], and [80]. Therefore, our results are novel. Equation (2.1) involves a critical exponent and lacks the compactness of embeddings in fractional Sobolev spaces, which poses significant challenges in investigating both the existence of weak solutions and the limit of the weak solution sequence. Compared with the results of Chen–Li–Yang and Ambrosio, for Theorem 2.2.4 when $m = 1$, our equation (2.1) features two potentials, V and Q, whereas the problems considered by Chen–Li–Yang [22] and Ambrosio [6] contain only a single potential V. Consequently, our result is new even in the case $m = 1$. Currently, there are no available results for $m \geq 2$ regarding equation (2.1). The works of Ambrosio [9] and Zheng et al. [80] investigated the multiplicity and concentration of solutions for problem (6). However, their problem lacks the critical term $|u|^{q_s^*-2}u$. Furthermore, their assumptions on the potential function V satisfy conditions (V_1) and (V_2), which differ from the assumptions on V in our work, leading them to employ a truncation method on the nonlinearity to study the existence of solutions. Therefore, our results are independent of theirs.

c) Regarding the third research topic:

- First, we prove the existence of weak solutions for equation (3.1). The initial result is Theorem 3.1.11. Under the conditions imposed on the potential V, we obtain a compact embedding from the solution space W into $L^q(\mathbb{R}^N, |x|^{-\gamma})$ for all $q \geq \dfrac{N}{s}$. From this, we establish the compactness of the (PS) sequence, thereby showing that its weak limit is a solution of the given problem. Theorem 3.1.11 extends the result of Thin [70]. In the case where the nonlinearity exhibits supercritical growth, we obtain Theorem 3.2.4 concerning the existence of weak solutions for equation (3.20). To establish this result, we introduce a modified function to reduce the problem to the critical case. Next, we establish an L^∞ estimate for the solutions and prove that the solution of the modified problem with the proposed modified function (3.25) is indeed a solution of problem (3.20).

The results of this chapter have been published in [75]. To the best of our knowledge, there are currently no results regarding the existence of weak solutions for the fractional p-Laplacian equation with supercritical exponential growth. Therefore, our results contribute to complementing and enriching the literature on the existence of solutions for multi-phase fractional equations with Trudinger–Moser type exponential growth.

4. Research methods and tools

This thesis employs variational methods to investigate the existence of weak solutions for the formulated equations. This is a highly effective research approach that has been widely utilized by mathematicians both domestically and internationally to study the existence of weak solutions for partial differential equations.

Furthermore, the thesis utilizes theoretical research methods such as analysis, synthesis, classification, and theoretical systematization. Specifically, we analyze and synthesize the results published by various authors within the research direction of the topic, thereby classifying the themes currently drawing attention and investigation from the domestic and international mathematical communities. Next, we systematize the topics and techniques implemented by these authors. Through this process, we identify open questions, issues requiring improvement, and novel problems to serve as the core research content of this thesis.

5. Thesis structure

The thesis consists of an introduction, three chapters, a conclusion, and a list of references.

Chapter 1: *Existence of weak solutions for multi-phase fractional problems in* $\mathbb{R}^N$. In this chapter, we present the existence of weak solutions for equations (1.18) and (1.36). The main results of this chapter are Theorem 1.2.5 and Theorem 1.3.3. The outcomes of this chapter have been published in [76] and [62, Theorem 5].

Chapter 2: *Concentration of solutions for multi-phase fractional Choquard equations in* $\mathbb{R}^N$ *with competing potentials.* The main results of this chapter are Theorem 2.2.4 and Theorem 2.2.5. In addition to the main results, we present in detail the technical lemmas utilized to establish these primary findings. The outcomes of this chapter have been published in [62].

Chapter 3: *Existence of weak solutions for multi-phase fractional Kirchhoff–Schrödinger type equations in* $\mathbb{R}^N$ *with exponential growth.* The main results of this chapter are Theorem 3.1.11 and Theorem 3.2.4. These findings have been published in [75].

In addition to being published in scientific journals, the main results of the thesis have been presented at:

- The Seminar of the Department of Analysis and Applied Mathematics, Faculty of Mathematics, University of Education, Thai Nguyen University.

- The International Conference on Geometry and Algebra, jointly organized by the University of Education - TNU and the Vietnam Institute for Advanced Study in Mathematics (VIASM).

Chapter 1

Existence of weak solutions for multi-phase fractional problems in $\mathbb{R}^N$

1.1 Preliminaries

This section presents the theoretical foundations, including the spaces L^p and $W^{s,p}(\mathbb{R}^N)$, the fractional p-Laplacian operator, and the Choquard convolution product. Concurrently, essential analytical tools such as the Fréchet derivative, the Nehari manifold, the Palais–Smale condition, the Mountain Pass Theorem, and the Trudinger–Moser inequality are introduced to facilitate the proofs of the existence of weak solutions in the subsequent chapters.

1.2 Existence of weak solutions for multi-phase fractional equations in $\mathbb{R}^N$ with exponential growth

In this section, we investigate the existence of solutions for the equation

$$\sum_{i=1}^{m}((-\Delta)_{p_i}^{s}u+\eta_1|u|^{p_i-2}u)=\eta_2^2[\frac{1}{|x|^{\mu}}*H_1(u)]h_1(u)+\eta_3h_2(u), \tag{1.18}$$

In equation (1.18), $H_j(t) = \int\limits_0^t h_j(t), j = 1, 2,\ 2 \leq p_1 = p < \cdots < p_m < +\infty,\ 0 < s < 1,\ N = ps, 0 < \mu < N$, and $\eta_1, \eta_2 > 0, \eta_3 > 0$ are constants. The two nonlinear functions h_1 and h_2 belong to $\mathfrak{O} := \{\mathfrak{h} : \mathbb{R} \to \mathbb{R}$ is a continuous function$\}$ and satisfy the following conditions:

($\mathfrak{h}_1$) The nonlinear function $\mathfrak{h}$ is continuous and $\mathfrak{h}(t) = 0$ on $(-\infty, 0]$. For each $q_1 \geq p_m,\ \ q_2 \geq 1$, and positive real numbers $a_1 > 0, a_2 > 0$ such as

$$\mathfrak{h}(t) \leq a_1|t|^{q_1-1} + a_2\Phi_{N,s}(\alpha_0|t|^{N/(N-s)})|t|^{q_2-1}$$

where $\Phi_{N,s_1}(y) = e^y - \sum_{j=0}^{j_p-2} \dfrac{y^j}{j!}, j_p = \min\{j \in \mathbb{N} : j \geq p\}$ and $\alpha_* < \alpha^*_{s,N}$, where $\alpha^*_{s,N} = N\left(\frac{2(Nw_N)^2\Gamma(p+1)}{N!}\sum_{k=0}^{\infty}\frac{(N+k-1)!}{k!}\frac{1}{(N+2k)^p}\right)^{\frac{s}{N-s}}$.

($\mathfrak{h}_2$) $\lim_{t\to 0^+}\frac{\mathfrak{h}(t)}{t^{p_m-1}} = 0$.

($\mathfrak{h}_3$) There exists $\theta > p_m$ such that $2\mathfrak{h}(t)t \geq \mathfrak{h}(t)t \geq \theta H(t) > 0$ for all $t > 0$, where $H(t) = \int_0^t \mathfrak{h}(\tau)d\tau$.

($\mathfrak{h}_4$) There exists a sufficiently large $\gamma_1 > 0$ such that $H(t) \geq \gamma_1|t|^\theta$ on $[0, \infty)$, where $\theta > p_m$ is the constant referred to in condition ($\mathfrak{h}_3$).

The weak solution of equation (1.18) is defined as follows:

Definition 1.2.1. A function $u \in W$ is called a weak solution of equation (1.18) if for every test function $\varphi \in W$, we have

$$\begin{aligned}
&\sum_{i=1}^m \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p_i-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp_i}}dxdy \\
&+ \sum_{i=1}^m \eta_1 \int_{\mathbb{R}^N} |u|^{p_i-2}u\varphi dx \\
&= \eta_2^2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H_1(u(y))}{|x-y|^\mu} h_1(u(x))\varphi(x)dxdy + \eta_3\int_{\mathbb{R}^N} h_2(u(x))\varphi(x)dx.
\end{aligned}$$

The energy functional $\mathcal{J}_{\eta_1,\eta_2,\eta_3} : W \to \mathbb{R}$ of equation (1.18) is given by

$$\begin{aligned}
&\mathcal{J}_{\eta_1,\eta_2,\eta_3}(u) \\
&= \sum_{i=1}^{m} \frac{1}{p_i} \|u\|^{p_i}_{\eta_1, W^{s,p_i}(\mathbb{R}^N)} - \frac{\eta_2^2}{2} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{H_1(u(y))H_1(u(x))}{|x-y|^{\mu}} dxdy - \eta_3 \int\limits_{\mathbb{R}^N} H_2(u)dx \\
&= \sum_{i=1}^{m} \frac{1}{p_i} \|u\|^{p_i}_{\eta_1, W^{s,p_i}(\mathbb{R}^N)} - \frac{\eta_2^2}{2} \int\limits_{\mathbb{R}^N} \mathcal{G}(u)(x) H_1(u(x))dx - \eta_3 \int\limits_{\mathbb{R}^N} H_2(u)dx,
\end{aligned}$$

where $\mathcal{G}(u)(x) = \int\limits_{\mathbb{R}^N} \frac{H_1(u(y))}{|x-y|^{\mu}} dy$. Then, according to Definition 1.2.1, we see that the weak solution of equation (1.18) is a critical point of the energy functional $\mathcal{J}_{\eta_1,\eta_2,\eta_3}$ in W.

Our first result is stated as follows:

Theorem 1.2.5. *Assume that* $(\mathfrak{h}_1) - (\mathfrak{h}_4)$ *are satisfied. Then equation* (1.18) *possesses a nontrivial, non-negative weak solution in* W.

1.3 Existence of weak solutions for multi-phase fractional Choquard equations with critical exponent

In this section, we investigate the existence of solutions for the equation

$$\sum_{i=1}^{m} (-\Delta)^s_{p_i} v + \gamma_1 \sum_{i=1}^{m} |v|^{p_i - 2} v = \gamma_2^2 \lambda \Big[\frac{1}{|x|^{\mu}} * F(v)\Big] \mathfrak{f}(v) + (v^+)^{q_s^* - 2} v^+ \quad (1.36)$$

where $\gamma_1 > 0, \gamma_2 > 0$ are constants. We assume that $\mathfrak{f}$ is a nonlinear function satisfying the following conditions:

(f_1) $\mathfrak{f} \in C(\mathbb{R}, \mathbb{R})$ and $\mathfrak{f}(t) = 0$ for all $t < 0$;

(f_2) $\lim_{t\to 0} \dfrac{\mathfrak{f}(t)}{t^{q-1}} = 0$ and there exists $a_0 > 0$ such that $|\mathfrak{f}(t)| \le a(1 + |t|^{\tau - 1})$ for all $t \ge 0$, where $q < \tau < \min\{\dfrac{q(N-s)}{N-qs}, \dfrac{q(N-\mu)}{N-qs}\}$;

(f_3) there exists $q_s^* > \theta > q$ such that $2\mathfrak{f}(t)t \ge \theta F(t)$ for all $t > 0$;

(f_4) the function $\dfrac{\mathfrak{f}(t)}{t^{\frac{q}{2}-1}}$ is increasing for $t > 0$. To study the existence of weak solutions for equation (1.36), we consider the energy functional

$$\mathcal{I}_{\gamma_1,\gamma_2}(v) = \sum_{i=1}^{m} \frac{1}{p_i}\Big(\iint\limits_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p_i}}{|x-y|^{N+p_i s}} dxdy + \int\limits_{\mathbb{R}^N} \gamma_1 |v(x)|^{p_i} dx\Big)$$
$$- \frac{\lambda\gamma_2^2}{2} \int\limits_{\mathbb{R}^N} \Big(\frac{1}{|x|^{\mu}} * F(v)\Big) F(v) dx + \frac{1}{q_s^*} \int\limits_{\mathbb{R}^N} (v^+)^{q_s^*} dx. \qquad (1.37)$$

For any arbitrary function $\phi \in W$, we have

$$\big\langle \mathcal{I}'_{\gamma_1,\gamma_2}(v), \phi \big\rangle = \sum_{i=1}^{m} \Big(\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p_i-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp_i}}\, dxdy$$
$$+ \gamma_1 \int_{\mathbb{R}^N} |v(x)|^{p_i-2} v(x)\phi(x)\, dx\Big)$$
$$- \lambda\gamma_2^2 \iint_{\mathbb{R}^{2N}} \frac{F(v(y))f(v(x))\phi(x)}{|x-y|^{\mu}}\, dxdy + \int_{\mathbb{R}^N} (v^+)^{q_s^*-2} v^+ \phi(x)\, dx.$$

Then the weak solution of (1.36) is defined as follows:

Definition 1.3.1. A function $v \in W$ is called a weak solution of equation (1.36) if for every test function $\phi \in W$, we have

$$\sum_{i=1}^{m} \Big(\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p_i-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp_i}}\, dxdy$$
$$+ \gamma_1 \int_{\mathbb{R}^N} |v(x)|^{p_i-2} v(x)\phi(x)\, dx\Big)$$
$$- \lambda\gamma_2^2 \iint_{\mathbb{R}^{2N}} \frac{F(v(y))f(v(x))\phi(x)}{|x-y|^{\mu}}\, dxdy + \int_{\mathbb{R}^N} (v^+)^{q_s^*-2} v^+ \phi(x)\, dx = 0.$$

From Definition 1.3.1, we see that a function $v \in W$ is a weak solution of (1.36) if $\big\langle \mathcal{I}'_{\gamma_1,\gamma_2}(v), \phi \big\rangle = 0$ for all $\phi \in W$. That is, $\mathcal{I}'_{\gamma_1,\gamma_2}(v) = 0$ in W^*. We denote by $\mathcal{M}_{\gamma_1,\gamma_2}$ the Nehari manifold associated with the energy functional $\mathcal{I}_{\gamma_1,\gamma_2}$

$$\mathcal{M}_{\gamma_1,\gamma_2} = \{v \in W \setminus \{0\} : \langle \mathcal{I}'_{\gamma_1,\gamma_2}(v), v\rangle = 0\}.$$

Definition 1.3.2. A function u is called a ground state solution of (1.36) if

$$\mathcal{I}_{\gamma_1,\gamma_2}(u) = \inf \big\{ \mathcal{I}_{\gamma_1,\gamma_2}(v) | v \in W \setminus \{0\}, \mathcal{I}'_{\gamma_1,\gamma_2}(v) = 0 \big\}.$$

Theorem 1.3.3. *Assume that conditions $(f_1) - (f_3)$ are satisfied. Then there exists a sufficiently large $\lambda_* > 0$ such that equation (1.36) possesses a nontrivial, non-negative weak solution for $\lambda \geq \lambda_*$.*

Conclusion of Chapter 1

In this chapter, we investigated the existence of weak solutions for multi-phase fractional equations with nonlinearities of Choquard type. The problem was considered in cases where the nonlinearities exhibit critical growth and Trudinger–Moser type exponential growth. The main results of this chapter are Theorem 1.2.5 and Theorem 1.3.3. To establish these results, we first demonstrated that the energy functional of the corresponding problem satisfies the geometric structure of the Mountain Pass Theorem. This implies the existence of a (PS) sequence with an estimated mountain-pass level. Next, by employing Lions-type compactness criteria, we showed that the weak limit of the (PS) sequence is a nontrivial weak solution of the given problem. The outcomes of this chapter have been published in [76] and [62, Theorem 5].

Chapter 2

Concentration of solutions for multi-phase fractional Choquard equations in $\mathbb{R}^N$ with competing potentials

2.1 Existence of ground state solutions

Motivated by multi-phase problems, in this chapter, we investigate the existence of weak solutions for an equation involving a multi-phase fractional operator in $\mathbb{R}^N$ with a Choquard nonlinearity and a critical exponent:

$$\sum_{i=1}^{m}(-\Delta)_{p_i}^{s}v+\sum_{i=1}^{m}V(\zeta x)|v|^{p_i-2}v \tag{2.1}$$
$$=\lambda\left(\frac{1}{|x|^{\mu}}*Q(\zeta y)F(v)\right)Q(\zeta x)\mathfrak{f}(v)+(v^{+})^{q_s^*-2}v^{+},\, v^{+}(x)=\max\{v(x),0\},$$

where $\lambda>0$ is sufficiently large, $F(t)=\int\limits_0^t\mathfrak{f}(\tau)d\tau$, $m\geq 2, 2\leq p=p_1<\cdots<p_m=q<\dfrac{N}{s}$, $s\in(0,1)$, $\zeta>0$ is a small parameter, and $0\leq\mu<qs$.

The potential function $V:\mathbb{R}^N\to\mathbb{R}$ is bounded and satisfies the Rabinowitz condition:

(V) $V \in C(\mathbb{R}^N)$ such that

$$+\infty > V_\infty = \liminf_{|x|\to\infty} V(x) \geq \inf_{x\in\mathbb{R}^N} V(x) = V_{\min} > 0.$$

Regarding the potential function Q, we assume that Q satisfies the following conditions:
(Q) $Q \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\limsup_{|x|\to\infty} Q(x) = Q_\infty$ and

$$Q_{\min} := \inf_{x\in\mathbb{R}^N} Q(x) > 0.$$

(VQ) $V(0) = V_{\min}$, $Q(0) = Q_{\max} := \sup_{x\in\mathbb{R}^N} Q(x) > Q_\infty$. In this case, $\mathcal{V} \cap \mathcal{Q} \neq \emptyset$, where

$$\mathcal{V} = \{x \in \mathbb{R}^N : V(x) = V_{\min}\}, \mathcal{Q} = \{x \in \mathbb{R}^N : Q(x) = Q_{\max}\}.$$

The nonlinear function $\mathfrak{f}$ satisfies conditions $(f_1) - (f_4)$.

We introduce the solution space for problem (2.1). Let $W^{s,p_i}_{V,\zeta}(\mathbb{R}^N)$ $(i = 1, \ldots, m)$ be the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|v\|_{W^{s,p_i}_{V,\zeta}(\mathbb{R}^N)} = \Big([v]^{p_i}_{s,p_i} + \|v\|^{p_i}_{V,p_i,\zeta}\Big)^{1/p_i}, \ \|v\|^{p_i}_{V,p_i,\zeta} = \int\limits_{\mathbb{R}^N} V(\zeta x)|v(x)|^{p_i} dx.$$

Set $W_\zeta = \cap_{i=1}^m W^{s,p_i}_{V,\zeta}(\mathbb{R}^N)$. Then W_ζ is a uniformly convex Banach space (analogous to [64], Lemma 10) equipped with the norm

$$\|v\|_{W_\zeta} = \sum_{i=1}^m \|v\|_{W^{s,p_i}_{V,\zeta}(\mathbb{R}^N)},$$

and hence W_ζ is a reflexive space.

To investigate the weak solutions of equation (2.1), we consider the energy functional $J_\zeta : W_\zeta \to \mathbb{R}$ given by

$$J_\zeta(v) = \sum_{i=1}^m \frac{1}{p_i} \|v\|^{p_i}_{W^{s,p_i}_{V,\zeta}(\mathbb{R}^N)} - \lambda \mathcal{K}_\zeta(v) - \frac{1}{q_s^*} \int\limits_{\mathbb{R}^N} (v^+)^{q_s^*} dx, \tag{2.3}$$

where $\mathcal{K}_\zeta(v) = \dfrac{1}{2} \int\limits_{\mathbb{R}^N} \Big[\dfrac{1}{|x|^\mu} * Q(\zeta y) F(v)\Big] Q(\zeta x) \mathfrak{f}(v) dx.$

The weak solution of equation (2.1) is defined as follows:

Definition 2.1.1. A function $v \in W_\zeta$ is called a weak solution of (2.1) if for every $\phi \in W_\zeta$, we have

$$\sum_{i=1}^{m} \int\limits_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p_i-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+p_i s}} dxdy$$
$$+\sum_{i=1}^{m} \int\limits_{\mathbb{R}^N} V(\zeta x)|v|^{p_i-2} v\phi dx - \lambda \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{Q(\zeta y)F(v(y))}{|x-y|^{\mu}} Q(\zeta x)\mathfrak{f}(v(x))\phi(x) dxdy$$
$$-\int\limits_{\mathbb{R}^N} (v^+)^{q_s^*-2} v^+ \phi dx = 0.$$

Hence, a weak solution v of (2.1) is a critical point of the energy functional J_ζ in W_ζ.

By employing variational methods, we establish the existence of a ground state solution for problem (2.1) for all sufficiently small ζ. The main result of this chapter is stated as follows:

Theorem 2.1.10. *Assume that $(V), (Q), (VQ)$, and $(f_1) - (f_3)$ are satisfied. Then there exists $\zeta_0 > 0$ such that problem (2.1) possesses a ground state solution for all $\zeta \in (0, \zeta_0)$.*

2.2 Concentration of solutions

Following the existence result of the ground state solution u_ζ for $\zeta \in (0, \zeta_0)$, in this section, we investigate the concentration phenomenon of solutions as $\zeta \to 0^+$ after an appropriate translation. The two main results are stated as follows:

Theorem 2.2.4. *Assume that $(V), (Q), (VQ)$, and $(f_1) - (f_5)$ are satisfied. Then there exists $\zeta_0 > 0$ such that equation (2.1) possesses a nontrivial positive solution for all sufficiently large λ and $\zeta \in (0, \zeta_0)$. We denote this solution by v_ζ and its global maximum by $x_\zeta \in \mathbb{R}^N$; then $\lim_{\zeta \to 0} V(x_\zeta) = V_{\min}$ and $\lim_{\zeta \to 0} Q(x_\zeta) = Q_{\max}$.*

Theorem 2.2.5. *Let v_ζ be the solution satisfying Theorem 2.2.4. Then there exists a sequence $\{\tilde{y}_\zeta\}$ such that $w_\zeta(x) := v_\zeta(x + \tilde{y}_\zeta)$ converges* *strongly in W to a ground state solution of*

$$\sum_{i=1}^{m}(-\Delta)_{p_i}^{s}v + V_{\min}\sum_{i=1}^{m}|v|^{p_i-2}v = \lambda Q_{\max}^2\left[\frac{1}{|x|^{\mu}} * F(v)\right]\mathfrak{f}(v) + |v|^{q_s^*-2}v$$

in $\mathbb{R}^N$ as $\zeta \to 0^+$.

Conclusion of Chapter 2

In this chapter, by employing variational methods, we establish the existence of weak solutions and the concentration phenomenon of solutions for the multi-phase fractional Choquard equation (2.1). The main results of this chapter are Theorem 2.2.4 and Theorem 2.2.5. The outcomes obtained in this chapter have been published in [62]. Theorem 2.2.4 demonstrates the existence of weak solutions for the non-autonomous problem (2.1) for sufficiently small ζ, and shows that the global maximum of the solution concentrates around the global minimum of the potential V and the global maximum of the potential Q. Theorem 2.2.5 states that the sequence of solutions for problem (2.1), after an appropriate translational change of variables, converges to a solution of the limiting problem. This result of ours is completely new. To prove the existence of solutions for the non-autonomous problem, we first show that the energy functional satisfies the geometric structure of the Mountain Pass Theorem. Consequently, there exists a (PS) sequence at the mountain-pass level C_ζ defined in (2.7). Next, we establish the compactness of the (PS) sequence when the mountain-pass level is strictly bounded above by the mountain-pass level of the limiting problem (1.18) for a sufficiently large λ. Thus, the weak solution of problem (2.1) is obtained as the limit of the (PS) sequence. The problem involves an attractive potential V and a repulsive potential Q, which introduces significant difficulties when establishing the compactness of the (PS) sequence. Furthermore, serious complications also arise when proving that the limit of the sequence of solutions for (2.1) converges to a solution of the limiting problem. Compared with the results of Chen–Li–Yang and Ambrosio, even in the case $m = 1$, our equation

(2.1) involves two potentials V and Q, whereas the problems considered by Chen–Li–Yang [22] and Ambrosio [6] only contain a single potential V. Therefore, our results are novel even for $m = 1$. Currently, there are no available results for $m \geq 2$ regarding equation (2.1). The works of Ambrosio [9], Zheng et al. [80] investigated the multiplicity and concentration of solutions for problem (6). However, their problem lacks the critical term $|u|^{q_s^*-2}u$. Moreover, the assumptions on the potential function V satisfy conditions (V_1) and (V_2), which differ fundamentally from the assumptions on V in our work, leading them to use a truncation method on the nonlinearity to study the existence of solutions. Consequently, our results are entirely independent of theirs.

Chapter 3

Existence of weak solutions for multi-phase fractional Kirchhoff–Schrödinger type equations in $\mathbb{R}^N$ with exponential growth

3.1 Multi-phase fractional Kirchhoff–Schrödinger type equations in $\mathbb{R}^N$ with critical exponential growth

In this chapter, we investigate the existence of weak solutions for the following singular multi-phase fractional Kirchhoff–Schrödinger equation:

$$\sum_{i=1}^{T} M_i\Big(\iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^{p_i} K_i(x-y)dxdy + \int_{\mathbb{R}^N} V(x)\,|u(x)|^{p_i}\,dx\Big) \\ \times (\mathcal{L}^s_{p_i} u(x) + V(x)\,|u|^{p_i-2}\,u) = \frac{f(x,u)}{|x|^{\gamma}}, \tag{3.1}$$

where $T \geq 1$, $N = p_1 s$, $\gamma \in [0, N)$, $s \in (0,1)$, and $2 \leq p = p_1 < \cdots < p_T < +\infty$. Here, V is a potential function satisfying conditions (V_1) and (V_2), or (V_1) and (V_3). The Kirchhoff functions $M_i(t)$ $(i = 1, \ldots, T)$ belong to the class $\mathfrak{M}$ of functions M satisfying:

(c_1) $M(t) \geq m_0 > 0$, $\forall t \in \mathbb{R}^+$,

(c_2) $M(t) \leq b_0 + b_1 t^{\alpha_1} + ... + b_m t^{\alpha_m}$, $t \geq 0$, with $b_i > 0$ $(i = 1, \dots, m)$ and $\alpha_m > ... > \alpha_1 \geq 1$. Consequently, $\mathcal{M}(t) = \int_0^t M(\tau) d\tau \leq b_0 t + b_1 \frac{t^{\alpha_1+1}}{\alpha_1+1} + ... + b_m \frac{t^{\alpha_m+1}}{\alpha_m+1}$.

(m_3) there exists $\theta > 1$ such that $\frac{M(t)}{t^{\theta-1}}$ is increasing on $(0, \infty)$.

The singular kernels K_i $(i = 1, \dots, T)$ satisfy the following conditions:

(C_1^i) $\eta K_i \in L^1(\mathbb{R}^N)$, where $\eta(x) = \min\{|x|^{p_i}, 1\}$.

(C_2^i) $\exists k_i > 0$ such that $K_i(x) \geq k_i |x|^{-(N+p_i s)}$, $\forall x \in \mathbb{R}^N \setminus \{0\}, i = 1, \dots, T$.

We assume that the continuous nonlinear function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ with critical exponential growth satisfies the following conditions:

(f_i) There exist constants $\alpha_0 \in (0, \beta_*)$ with $0 < \beta_* < \alpha_*$, and $b_1, b_2 > 0$ such that $|f(x,t)| \leq b_1 |t|^{p-1} + b_2 \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $\Phi_{N,s}(y) = e^y - \sum_{i=0}^{j_p-2} \frac{y^j}{j!}$ and $j_p = \min\{j \in \mathbb{N} : j \geq p\}$.

(f_{ii}) There exists $\mu > \max\{p_T\theta, p_T(\alpha_m + 1)\}$ such that $f(x,t) - \mu F(x,t) \geq -\rho |t|^{N/s}$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $\rho \geq 0$ and satisfies $\frac{\rho A_{N/s,\gamma}^{-N/s}}{\mu} < \frac{m_0}{p\theta} - \frac{m_0}{\mu}$, where $\theta > 1$ is the constant given in condition (m_3).

(f_{iii}) There exists a constant $B > 0$ satisfying $\lim_{t\to 0} \frac{F(x,t)}{|t|^{N/s}} < B$ uniformly with respect to $x \in \mathbb{R}^N$ such that $\frac{m_0 s}{N} - B A_{N/s,\gamma}^{-N/s} > 0$.

(f_{iv}) There exists $\gamma_1 > 0$ such that $F(x,t) \geq \gamma_1 |t|^{\mu}$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $\mu > \max\{p_T\theta, p_T(\alpha_m + 1)\}$ is the constant given in condition (f_{ii}).

The potential function V is a continuous function satisfying the following conditions:

(V_1) $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$;

(V_2) For each $c > 0$, we have meas $\left(\{x \in \mathbb{R}^N : V(x) \leq c\}\right) < \infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^N$. We assume that V satisfies the following additional conditions besides (V_1) and (V_2):

(V_3) $\exists h > 0$ such that $\lim_{|y|\to\infty} \text{meas}(\{x \in B_h(y) : V(x) \leq c\}) = 0$ for all $c > 0$.

Before presenting the main results, we establish the solution space for the problem. Let $\mathcal{W}^{s,p_i}_{K_i}$ $(i = 1, \dots, T)$ denote the fractional Sobolev space defined by: $\mathcal{W}^{s,p_i}_{K_i}(\mathbb{R}^N) := \{v \in L^{p_i}(\mathbb{R}^N) : [v]_{s,K_i,p_i} < \infty\}$, where $[v]_{s,K_i,p_i} = \Big(\iint\limits_{\mathbb{R}^{2N}} |v(x) - v(y)|^{p_i} K_i(x-y)dxdy\Big)^{1/p_i}$. Note that $\mathcal{W}^{s,p_i}_{K_i}(\mathbb{R}^N)$ is a uniformly convex Banach space (as shown in [64]) equipped with the norm $||v||_{\mathcal{W}^{s,p_i}_{K_i}(\mathbb{R}^N)} = \Big(||v||^{p_i}_{L^{p_i}(\mathbb{R}^N)} + [v]^{p_i}_{s,K_i,p_i}\Big)^{1/p_i}$. Let $\mathcal{W}_i$ $(i = 1, \dots, T)$ be the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $||v||_{\mathcal{W}_i} = \Big([v]^{p_i}_{s,p_i} + ||v||^{p_i}_{p_i,V}\Big)^{1/p_i}$, $||v||^{p_i}_{p_i,V} = \int\limits_{\mathbb{R}^N} V(x)|v(x)|^{p_i}dx$.

Then, $\mathcal{W}_i$ is a uniformly convex Banach space (according to [64], Lemma 10), and $\mathcal{W}$ is a reflexive space. An example of the singular kernel K_i $(i = 1, \dots, T)$ is given by $K_i(x) = |x|^{-(N+p_is)}$. When $K_i(x) = |x|^{-(N+p_is)}$, we utilize the notation $W^{s,p_i}(\mathbb{R}^N)$ instead of $\mathcal{W}^{s,p_i}_{K_i}(\mathbb{R}^N)$ and $[v]_{s,p_i}$ instead of $[v]_{s,K_i,p_i}$. Set $\mathcal{W} = \cap_{i=1}^T \mathcal{W}_i$.

The weak solution of equation (3.1) is defined as follows:

Definition 3.1.1. A function $u \in \mathcal{W}$ is called a weak solution of equation (3.1) if for every test function $\varphi \in \mathcal{W}$, we have

$$\sum_{i=1}^T M_i(||u||^{p_i}_{\mathcal{W}_i})\mathcal{A}_i(u,\varphi) = \int_{\mathbb{R}^N} \frac{f(x,u)\varphi(x)}{|x|^\gamma}dx,$$

where

$$\begin{aligned}\mathcal{A}_i(u,\varphi) = &\iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p_i-2}(u(x) - u(y))(\varphi(x) - \varphi(y))K_i(x-y)dxdy \\ &+ \int_{\mathbb{R}^N} V(x)|u|^{p_i-2}u\varphi dx.\end{aligned}$$

To establish the main results in this section, we consider the energy functional

$$J(u) = \sum_{i=1}^T \frac{1}{p_i}\mathcal{M}_i(\|u\|^{p_i}_{\mathcal{W}_i}) - \int_{\mathbb{R}^N} \frac{F(x,u)}{|x|^\gamma}dx.$$

According to Definition 3.1.1, we see that u is a critical point of the energy functional J in $\mathcal{W}$ if and only if u is a weak solution of equation (3.1).

The main result is presented in the following theorem:

Theorem 3.1.11. *Suppose that V satisfies conditions (V_1) and (V_2) or (V_1) and (V_3), f satisfies conditions (f_i)–(f_{iv}), and the Kirchhoff functions M_i $(i = 1, \ldots, T)$ satisfy (m_3), (c_1), and (c_2). Then, for any $\gamma_1 \geq \max\{a, \gamma_0^*\}$, problem (3.1) possesses a nontrivial solution, where*

$$a = \sum_{i=1}^{T} \frac{1}{p_i} \left(b_0(\mathcal{A}_{\mu,\gamma} + 1)^{p_i} + \sum_{j=1}^{m} \frac{b_j}{\alpha_j + 1} (\mathcal{A}_{\mu,\gamma} + 1)^{p_i(\alpha_j+1)} \right),$$

$$\gamma_0^* = \frac{pa}{\mu} \Big[\Big(\frac{\alpha_*}{4p'\alpha_0} (1 - \frac{\gamma}{N}) \Big)^{\frac{N-s}{s}} \frac{\min\{k_1, V_0\}}{2a \left(1 - \frac{pa}{\mu}\right)} \Big(\frac{m_0}{p\theta} - \frac{m_0}{\mu} - \frac{\rho}{\mu} A_{N/s,\gamma}^{-N/s} \Big) \Big]^{\frac{p-\mu}{p}}$$

and $p' = \dfrac{p}{p-1}$ is the conjugate exponent of p.

3.2 Multi-phase fractional Kirchhoff–Schrödinger type equations in $\mathbb{R}^N$ with supercritical exponential growth

We investigate the existence of solutions for (3.1) when $\gamma = 0$ and $f(x, t) = h(t)e^{\beta_0 |t|^\tau}$ $(t \in \mathbb{R})$, where $\beta_0 > 0$ and $\tau \geq \dfrac{N}{N-s}$. Specifically, in this section, we establish the existence of weak solutions for the equation

$$\sum_{i=1}^{T} M_i \Big(\iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p_i} K_i(x - y) dxdy + \int_{\mathbb{R}^N} V(x) |u(x)|^{p_i} dx \Big) \times (\mathcal{L}_{p_i}^s u(x) + V(x) |u|^{p_i - 2} u) = h(t)e^{\beta_0 |t|^\tau}. \tag{3.20}$$

We assume that h is a continuous function satisfying the following conditions:

(h_1) $h(t) \equiv 0$ for all $t \in (-\infty, 0]$ and $\lim_{t \to 0^+} \dfrac{h(t)}{t^{p_T - 1}} = 0$;

(h_2) There exists $\mu > \max\{\theta p_T, (\alpha_m + 1)p_T\}$ such that $0 < \mu H(t) \leq h(t)t$ for all $t \in (0, \infty)$, where $H(t) = \int\limits_0^t h(\tau) d\tau$ and θ is given in condition (m_3).

(h_3) There exist $\delta \in (0, \dfrac{N}{N-s})$, $\gamma > 0$, and $C > 0$ such that $0 \leq h(t) \leq Ce^{\gamma |t|^\delta}$ for all $t \in [0, \infty)$.

(h_4) There exists a sufficiently large constant $\gamma_1 > 0$ such that $H(t) \geq \gamma_1 |t|^\mu$ for all $t \geq 0$.

Motivated by the results in [5, 67], for each $R \geq 1$, we consider the following function:

$$f_R(t) = \begin{cases} 0, & t \leq 0 \\ h(t)e^{\beta_0 t^\tau}, & 0 \leq t \leq R \\ h(t)e^{\beta_0 R^{\tau-\delta} t^\delta}, & t \geq R \end{cases}. \tag{3.22}$$

We deduce that f_R satisfies conditions (f_i), (f_{ii}) with $\rho = 0$, (f_{iii}), and (f_{iv}). When $\gamma = 0$, a is replaced by a^* as follows:

$$a^* = \sum_{i=1}^{T} \frac{1}{p_i} \left(b_0(\mathcal{A}_{\mu,0} + 1)^{p_i} + \sum_{j=1}^{m} \frac{b_j}{\alpha_j + 1} (\mathcal{A}_{\mu,0} + 1)^{p_i(\alpha_j+1)} \right).$$

In the constant γ_0^*, we set $\rho = 0$ and $\gamma = 0$; then γ_0^* is replaced by

$$\gamma_0^{**} = \frac{pa^*}{\mu} \Big[\Big(\frac{\alpha_*}{4p'\alpha_0} \Big)^{\frac{N-s}{s}} \frac{\min\{k_1, V_0\}}{2a^* \left(1 - \frac{pa^*}{\mu}\right)} \Big(\frac{m_0}{p\theta} - \frac{m_0}{\mu} \Big) \Big]^{\frac{p-\mu}{p}}.$$

Definition 3.2.1. A function $u \in \mathcal{W}$ is called a weak solution of equation (3.20) if for every test function $\varphi \in \mathcal{W}$, we have

$$\sum_{i=1}^{T} M_i(||u||_{\mathcal{W}_i}^{p_i}) \mathcal{A}_i(u, \varphi) = \int_{\mathbb{R}^N} f(x, u)\varphi(x)dx,$$

where $f(x, u) = h(u(x))e^{\beta_0 |u(x)|^\tau}$.

Since the nonlinearity in equation (3.20) exhibits supercritical exponential growth, the embedding is no longer compact and there is no Trudinger–Moser type inequality available for the supercritical case. Consequently, applying variational methods directly to establish the existence of weak solutions for the problem becomes remarkably challenging. To overcome this difficulty, we introduce a modified function to reduce the problem to the critical case. Then, we can consider the modified problem as follows:

$$\sum_{i=1}^{T} M_i \Big(\iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p_i} K_i(x - y)dxdy + \int_{\mathbb{R}^N} V(x) |u(x)|^{p_i} dx \Big) \\ \times (\mathcal{L}_{p_i}^s u(x) + V(x) |u|^{p_i - 2} u) = f_R(u). \tag{3.25}$$

Since $h(t) = 0$ for all $t \in (-\infty, 0]$, it follows that all solutions of (3.20) and (3.25) are non-negative. The weak solution of (3.25) is understood in the sense of Definition 3.2.1 by replacing $f(x,u)$ with $f_R(u)$. Then we have:

Theorem 3.2.2. *Assume that V satisfies conditions (V_1) and (V_2) or (V_1) and (V_3), h satisfies conditions $(h_1) - (h_4)$, and the Kirchhoff functions M_i $(i = 1, \dots, T)$ satisfy (m_3), (c_1), and (c_2). Then there exists $\gamma^* = \max\{a^*, \gamma_0^{**}\} > 0$ such that problem (3.25) possesses a nontrivial, non-negative weak solution in $\mathcal{W}$ for any $\gamma_1 \geq \gamma^*$.*

Fix $\varepsilon_0 > 0$ with ε_0 being close to 0. We denote

$$\gamma_0^{***} = \frac{pa^*}{\mu}\left(\frac{\left(\frac{\alpha_*}{4(1+\varepsilon_0)\alpha_0}\right)^{\frac{N-s}{s}}\min\{k_1, V_0\}}{a^*(1-\frac{pa^*}{\mu})}\right)^{\frac{p-\mu}{p}}.$$

Next, we show that the solution u obtained from Theorem 3.2.2 is indeed a solution of equation (3.20):

Theorem 3.2.4. *Suppose that V satisfies conditions (V_1) and (V_2) or (V_1) and (V_3), h satisfies conditions $(h_1) - (h_4)$, and the Kirchhoff functions M_i $(i = 1, \dots, T)$ satisfy (m_3), (c_1), and (c_2). For any $\tau \geq \dfrac{N}{N-s}$, there exist $\beta_{**}(\tau) = \dfrac{1}{R^{\tau-\delta}} > 0$ and $\gamma^{***} = \max\{a^*, \gamma_0^{**}, \gamma_0^{***}\} > 0$ such that problem (3.20) possesses a nontrivial solution $u \in \mathcal{W}$ for all $\gamma_1 \geq \gamma^{***}$ and $\beta_0 \in (0, \beta_{**}(\tau))$.*

Conclusion of Chapter 3

In this chapter, we investigated the existence of solutions for multi-phase fractional Kirchhoff-type equations where the nonlinearities exhibit critical and supercritical exponential growth. The first primary result is Theorem 3.1.11. This equation has been investigated here for the first time; hence, our results are entirely novel. Specifically, this outcome extends the results in [70] to multi-phase fractional Kirchhoff-type problems. The main difficulty arises from the presence of the Kirchhoff functions

M_i $(i = 1, \ldots, T)$, which required us to carefully adapt the analytical calculations from [70] to suit the structure of this new equation. Based on the assumptions on the potential V, we establish a compact embedding from the solution space $\mathcal{W}$ into $L^q(\mathbb{R}^N, |x|^{-\gamma})$ for any $q \geq \dfrac{N}{s}$. Consequently, we are able to prove the compactness of the (PS) sequence and show that its weak limit is a solution of the given problem. For the case where the nonlinearity exhibits supercritical exponential growth, we obtained Theorem 3.2.4. Since the Trudinger–Moser type inequality is unavailable in fractional Sobolev spaces for the supercritical case, we introduced a novel modified function to reduce the problem to the critical framework. Subsequently, we established L^∞-estimates for the solutions and demonstrated that the solution of the modified problem with the proposed truncation (3.25) is indeed a solution of the original problem (3.20). The outcomes of this chapter have been published in [75]. Our results contribute to complementing and enriching the study of the existence of solutions for multi-phase fractional equations with Trudinger–Moser type exponential growth.

General Conclusion

In this dissertation, we investigate the existence and concentration phenomena of solutions for several classes of elliptic equations involving fractional p-Laplace operators. Specifically, we focus on multi-phase fractional problems.

The main results of the dissertation include:

1. The primary results of Chapter 1 are Theorem 1.2.5 and Theorem 1.3.3, which establish the existence of weak solutions for multi-phase fractional equations with Choquard-type nonlinearities. The nonlinear functions considered exhibit subcritical growth and Trudinger–Moser type critical exponential growth.
2. The primary results of Chapter 2 are Theorem 2.2.4 and Theorem 2.2.5 concerning the existence, concentration of solutions, and the asymptotic behavior (limit) of the solution sequence for the non-autonomous problem (2.1).
3. The primary results of Chapter 3 consist of Theorem 3.1.11 and Theorem 3.2.4. Theorem 3.1.11 addresses the existence of solutions for the singular multi-phase fractional Kirchhoff-type equation (3.1) when the nonlinearity exhibits Trudinger–Moser type exponential growth. Subsequently, we establish the existence of weak solutions for equation (3.20) where the nonlinearity features supercritical exponential growth.

We propose several directions for future research:

1. Proving the existence of solutions for (2.1) in the case where λ is small.
2. Investigating the existence of solutions for systems of multi-phase fractional equations.

List of Author's Publications Related to the Dissertation

1. [62] Pham T. Q., Nguyen T. V. (2026), Concentration phenomena for anisotropic fractional Choquard equations and potential competition, Bulletin of Mathematical Sciences, Vol. 16, No. 1, Article number: 2550018 (48 pages).
2. [75] Trang P. Q., Ahmad B., Thin N. V. (2026), On fractional anisotropic Kirchhoff-Schrödinger type equations in $\mathbb{R}^N$, Discrete and Continuous Dynamical Systems - S. doi: 10.3934/dcdss.2026112.
3. [76] Trang P. Q. (2025), The existence of weak solution to fractional anisotropic equation in $\mathbb{R}^N$ with exponential growth, *TNU Journal of Science and Technology*, 230(06), pp. 487 - 494.

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