THAI NGUYEN UNIVERSITY
**UNIVERSITY OF EDUCATION**

**NGUYEN DUY PHUONG**

# ON UNIQUE POLYNOMIALS AND UNIQUE RANGE SETS FOR $L$-FUNCTIONS AND MEROMORPHIC FUNCTIONS

Major: Mathematical Analysis
Code: 9460102

## SUMMARY OF Ph.D. DISSERTATION IN MATHEMATICS

**THAI NGUYEN - 2026**

The dissertation was completed at:

**UNIVERSITY OF EDUCATION - THAI NGUYEN UNIVERSITY**

**Scientific Supervisors:**
1. Dr. Vu Hoai An
2. Dr. Le Quang Ninh

**Reviewer 1:** ........................................
**Reviewer 2:** ........................................
**Reviewer 3:** ........................................

The dissertation will be defended before the University-level Dissertation Assessment Council held at: University of Education - Thai Nguyen University at ... (time) on ... (date) ... (month), 2026.

*The dissertation can be found at:*
- National Library of Vietnam;
- Digital Center - Thai Nguyen University;
- Library of University of Education - Thai Nguyen University.

# Introduction

## 1. Research history and motivation

In the 1920s, inheriting the idea of the existence of roots from the Fundamental Theorem of Algebra for polynomials over the complex field, the Finnish mathematician R. Nevanlinna established one of the most profound and influential theories in Complex Analysis, namely the value distribution theory of meromorphic functions on $\mathbb{C}$, widely known today as Nevanlinna Theory. The value distribution theory marked a breakthrough in complex analysis by shifting the focus from investigating the existence of roots of polynomials to exploring the value distribution of meromorphic functions. The core of this theory lies in two fundamental theorems. Specifically, the First Fundamental Theorem serves as a deep generalization of the Fundamental Theorem of Algebra to describe the uniform value distribution of non-constant meromorphic functions on the complex plane $\mathbb{C}$. Building on that foundation, the Second Fundamental Theorem describes the influence of derivatives on this value distribution process. During its development, Nevanlinna Theory has attracted the attention and contributions of many mathematicians, such as H. Cartan

[10], who extended the theory to linear combinations and holomorphic curves in projective spaces, creating a geometric foundation for the studies of H. Weyl [62], L. Ahlfors [1], and P. Griffiths [9, 19], ...and it has many applications in various fields of Mathematics.

In this context, it is important to mention the early contributions of Vietnamese mathematics to this field. The first milestone was established by L. V. Thiem [37] regarding the inverse problem of value distribution theory. He proved the existence of meromorphic functions with prescribed ramification indices through the theory of Riemann surfaces.

One of the applications of Nevanlinna theory is the uniqueness problem for meromorphic functions, which aims to find conditions for two meromorphic functions to be identical or linearly related. The starting result for this theory is R. Nevanlinna's 5-point theorem (1926), asserting that a meromorphic function is uniquely determined by the inverse images of five distinct values.

To establish tools for investigating the uniqueness problem, mathematicians have developed a standardized notational framework to describe the coincidence of inverse images of two meromorphic functions. Specifically, these tools are established as follows:

Let $f$ be a non-constant meromorphic function on $\mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$, denote by $E_f(a)$ (respectively $\overline{E}_f(a)$) the set of $a$-points of $f$ where each $a$-point is counted with its multiplicity (respectively the set of distinct $a$-points of $f$). We denote

$$E_f(S) = \bigcup_{a \in S} E_f(a); \quad \overline{E}_f(S) = \bigcup_{a \in S} \overline{E}_f(a).$$

Two meromorphic functions $f, g$ are said to share the value $a$ counting multiplicities (ignoring multiplicities) if $E_f(a) = E_g(a)$ (respectively $\overline{E}_f(a) = \overline{E}_g(a)$). Two meromorphic functions $f, g$ are said to share the set $S$ CM (IM) if $E_f(S) = E_g(S)$ (respectively $\overline{E}_f(S) = \overline{E}_g(S)$).

From Nevanlinna's results, the problem was developed towards set sharing. F. Gross [20] studied the structure of the inverse images of sets and proposed the concept of unique range sets. The results in this research direction were systematized by C.-C. Yang and H.-X. Yi [67] into basic identity criteria for meromorphic functions through the uniqueness problem for unique range sets and bi-unique range sets. To minimize the number of elements in shared sets, I. Lahiri [35] introduced the weighted sharing method, which was later developed by A. Banerjee [6] and many other authors.

Simultaneously, the uniqueness problem has also been extended to other spaces and structural objects. On the $p$-adic number field, the foundation of Nevanlinna theory was established by H. H. Khoai [27], and the value sharing problem was further developed by him and A. Escassut [12]. For several variables, the theory was extended by H. Fujimoto [16] and M. Ru [56] for holomorphic curves and meromorphic mappings.

Contributing to the general development of the uniqueness problem, Vietnamese mathematicians have achieved many results for various classes of objects. Notable examples include the research of D. D. Thai and S. D. Quang [61] on meromorphic mappings of several variables with bounded multiplicity conditions, T. V. Tan [11] on the moving target sharing problem, T. T. H. An [2] on the uniqueness of differential polynomials sharing small functions, and H. T. Phuong [54] on Nevanlinna theory and the uniqueness problem for holomorphic curves on annuli sharing hypersurfaces. These studies, along with the works of other authors, have contributed to the domestic development of the theory.

In 1926, R. Nevanlinna (see [51]) proved that if two meromorphic functions share five distinct values IM, they are identically equal. This result shows that two meromorphic functions are uniquely determined by the IM inverse images of five distinct points. Since then, this research direction has attracted many domestic and foreign authors, such as the studies by G. Gundersen [22, 23] on the multiplicity of shared values, or E. Mues [50, 49] extending the problem to value sharing related to derivatives, ...

In 1977, F. Gross [21] developed the uniqueness problem in a new direction. Instead of considering the inverse images of individual points, he proposed the idea of considering the inverse images of a set of points $S \subset \mathbb{C} \cup \{\infty\}$. He posed two research problems for meromorphic functions:

*(i) Find unique range sets for meromorphic functions, i.e., find sets* $S \subset \mathbb{C} \cup \{\infty\}$ *such that any two meromorphic functions sharing the set* $S$ *CM or IM are identically equal;*

*(ii) Find bi-unique range sets for meromorphic functions, i.e., find pairs of sets* $S_1, S_2 \subset \mathbb{C} \cup \{\infty\}$ *such that any two meromorphic functions sharing the sets* $S_1$ *and* $S_2$ *CM or IM are identically equal.*

H.-X. Yi (see [69, 67]) was the first to provide an answer to Gross's

Problem. Since then, the results achieved in this research direction have been very rich and profound (see [16, 17, 38, 14, 28, 31, 30, 6, 41, 57, 2, 55, 59, 66]).

These research works can be divided into two basic themes:

1. Finding unique range sets and bi-unique range sets for meromorphic functions or special classes of meromorphic functions with the minimum possible number of elements, under the conditions that the functions or their differential polynomials share one or more elements/sets;

2. Establishing characterizations of unique range sets and bi-unique range sets.

Parallel to the above topics, extending the problem to the class of $L$-functions is becoming a specialized research direction. J. Steuding [60] established the value distribution theory foundation for the extended Selberg class, paving the way for the uniqueness theorems of B. Q. Li [45, 46]. Applying techniques concerning differential polynomials, multi-variable mappings, and set-sharing conditions—areas where Vietnamese mathematicians have made significant contributions as mentioned above—to $L$-functions is yielding promising results. These techniques serve as key tools, allowing the extension of uniqueness theorems to set-sharing conditions and moving targets, confirming the instrumental value of complex analysis methods in exploring the structure of $L$-functions, and establishing a solid theoretical basis for the research results in this dissertation.

In this dissertation, we will focus on the first theme for meromorphic functions or $L$-functions. Following this research direction, with the goal of finding unique range sets, in 2018, Q.-Q. Yuan, X.-M. Li, and H.-X. Yi [70] indicated a form of a unique range set CM for an $L$-function and a meromorphic function. The authors proved:

**Theorem 1.** *Let $f$ be a non-constant meromorphic function with finitely many poles and $L$ be a non-constant $L$-function. Let $Y_{n,m}(z) = z^n + az^m + b$, with $a, b \neq 0$, where $m, n$ are positive integers, $n > 2m + 4$, $\gcd(n, m) = 1$, and let $S_Y = \{z : Y_{n,m}(z) = 0\}$. Then, if $E_f(S_Y) = E_L(S_Y)$, we have $f = L$.*

Theorem 1 provides us with a form of a unique range set for a non-constant $L$-function and a non-constant meromorphic function with finitely many poles $S$ consisting of 7 elements (when choosing $m = 1, n = 7$). The polynomial $Y_{n,m}(z)$ in Theorem 1 is called a Yi-type polynomial.

In 2023, H. H. Khoai, V. H. An, and N. D. Phuong [34] designed a new polynomial form $P_K(z)$ different from the $Y_{n,m}(z)$ type polynomial and introduced a class of unique range sets for an $L$-function and a meromorphic function with finitely many poles consisting of 5 elements. Specifically, the authors considered $P_K(z) \in \mathbb{C}[z]$ of the form:

$$\begin{aligned} P_K(z) &= (n+m+1)\left(\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i} z^{n+m+1-i} a^i\right) + 1 \\ &= Q_K(z) + 1, \end{aligned}$$

with the conditions $a \neq 0$, $Q_K(a) \neq -1$, and $P_K(a) \neq 0$. Denote

$$S_K = \{z : P_K(z) = 0\}. \tag{1}$$

The authors proved the following results:

**Theorem 2** ([34]). *Let $f$ be a non-constant meromorphic function with finitely many poles, $L$ be a non-constant $L$-function, and $S_K$ be defined in* (1). *Then $L = f$ if $E_L(S_K) = E_f(S_K)$ and $n \geq m + 2$.*

**Theorem 3** ([34]). *Let $f$ be a non-constant meromorphic function, $L$ be a non-constant $L$-function, and $S_K$ be defined in* (1). *Then $L = f$ if $E_L(S_K) = E_f(S_K)$ and $n \geq 2$, $m \geq 2$, $n + m \geq 8$.*

Theorem 2 shows the existence of a unique range set for a meromorphic function with finitely many poles and an $L$-function consisting of 5 elements (when choosing $m = 1, n = 3$), reduced from 7 elements compared to Theorem 1. At the same time, this theorem also relaxes the degree condition ($n \geq m + 2$) and completely removes the hypothesis $\gcd(n, m) = 1$. Theorem 3 shows the existence of a unique range set for a general meromorphic function and an $L$-function consisting of 9 elements (when choosing $n + m = 8$). Simultaneously, this theorem completely removes the "finitely many poles" hypothesis to apply to any non-constant meromorphic function, with the trade-off that the degree conditions are tighter ($n \geq 2$, $m \geq 2$, and $n + m \geq 8$).

In 2023, based on the class of polynomials $P(z)$ satisfying the Fujimoto conditions, A. Banerjee and A. Kundu [8] asserted the identity of an $L$-function and a meromorphic function through sharing a set with weight 2, a generalization of the result by H. H. Khoai, V. H. An, and N. D. Phuong. Specifically, the authors proved the theorem:

**Theorem 4.** *Let $P(z) = a(z - a_1) \cdots (z - a_q)$ having the derivative $P'(z) = aq(z - d_1)^{q_1} \cdots (z - d_k)^{q_k}$ satisfying the condition $P(d_i) \neq P(d_j)$ for all $1 \leq i < j \leq k$ and being a strong uniqueness polynomial* *for meromorphic functions. Denote* $S = \{z : P(z) = 0\}$*. Then for a non-constant meromorphic function* $f$ *and a non-constant* $L$*-function* $L$*, if* $\min\{q_1, q_2\} \geq 2$ *when* $k = 2$*,* $q \geq 2k+4$*, and* $E_f(S, 2) = E_L(S, 2)$*, we have* $f = L$*.*

It can be seen that the polynomial class in Theorem 4 is more general than the polynomial class in Theorem 3, and the condition $E_L(S_K) = E_f(S_K)$ will imply the condition $E_f(S, 2) = E_L(S, 2)$; therefore, Theorem 4 is a significant improvement of Theorem 3.

Regarding the study of bi-unique range sets, especially a pair consisting of a finite set $S$ and the set of poles $\{\infty\}$, it often leads to a linear dependence relation $f = cg$. The existence of this relation for the set $S$ generated by the polynomial $Y_{n,m}(z)$ was proved by P. Li and C.-C. Yang [38] and H.-X. Yi [69]. Subsequently, H. Fujimoto [16] introduced general conditions for strong uniqueness polynomials to explain the appearance of the constant $c$. Extending this problem to $L$-functions, Q.-Q. Yuan, X.-M. Li, and H.-X. Yi [70] also obtained a similar linear dependence relation and used the hypothesis $\gcd(n, m) = 1$ to deduce the absolute identity $f \equiv g$.

Following this approach, in 2021, H. H. Khoai, V. H. An, and L. Q. Ninh [32] studied the problem of sharing inverse images of a pair of sets $S, T$ generated by the polynomial $Y_{n,m}(z)$, pointing out the existence of the linear dependence relation $f = hL$ or absolute identity when $\gcd(n, m) = 1$ through the following theorem:

**Theorem 5.** *Let* $m, n$ *be positive integers such that* $n \geq 2m + 8$*, and let* $P(z), Q(z)$ *be* $Y_{n,m}$*-type polynomials with different coefficients. Let* $S$ *and* $T$ *be the sets of zeros of* $P(z)$ *and* $Q(z)$*, respectively. Suppose* $f$

*is a non-constant meromorphic function with finitely many poles and $L$ is a non-constant $L$-function satisfying $\overline{E}_L(S) = \overline{E}_f(S)$. Then:*

*(i) There exists a non-zero constant $h$ such that $f = hL$.*

*(ii) If $\gcd(n, m) = 1$, then $f = L$.*

*(iii) In particular, if $f$ is a non-constant $L$-function, then $f = L$ and $P = Q$.*

It can be seen that Theorem 5 gives us a bi-unique range set form for a non-constant meromorphic function with finitely many poles and a non-constant $L$-function as a pair of sets of zeros of $Y_{n,m}(z)$-type polynomials. Furthermore, this theorem also provides an algebraic condition for an $L$-function and a meromorphic function with finitely many poles to be expressed linearly in terms of each other. This theorem also opens up a research direction on bi-unique range sets for meromorphic functions.

Regarding the uniqueness problem related to the condition that differential polynomials share one or more sets, it is considered to originate from the conjecture of W. K. Hayman [25] (see also [24]) on the value distribution of differential polynomials. In 1997, C.-C. Yang and X. H. Hua [66] studied differential polynomials of the form $f^n f'$ and proved the following theorem:

**Theorem 6** ([66]). *Let $n \geq 11$ be an integer and $a$ be a non-zero complex number. Suppose two non-constant meromorphic functions $f$ and $g$ satisfy that the differential polynomials $f^n f'$ and $g^n g'$ share the value $a$ CM. Then, either $f = dg$ with $d^{n+1} = 1$, or $f(z) = c_1 e^{cz}$ and*

$g(z) = c_2 e^{-cz}$, *where* $c, c_1, c_2$ *are constants and satisfy* $(c_1c_2)^{n+1}c^2 = -a^2$.

It can be seen that Theorem 6 provides a strict analytic criterion for the uniqueness of meromorphic functions through the structure of differential polynomials. The use of high powers ($n \geq 11$) reduced the condition from sharing multiple values to sharing only a single value $a \neq 0$ (CM), while precisely classifying the two functions into either proportional or exponential forms. This result created an important theoretical premise to extend the set sharing problem to higher-order derivatives and more complex objects such as $L$-functions in subsequent studies. Notable examples include J. Ojeda [52] with applications of W. K. Hayman's conjecture in the $p$-adic setting, [53] on differential polynomials sharing finite sets; M. L. Fang and W. Hong [13] in optimizing the degree of polynomials, or I. Lahiri [35] with the weighted sharing method. Furthermore, the contributions of H. Fujimoto [17] and A. Banerjee [6] also played important roles.

For the class of $L$-functions, in 2017, F. Liu, X.-M. Li, and H.-X. Yi [48] extended the problem to higher-order derivatives and proved:

**Theorem 7** ([48]). *Let* $f$ *be a non-constant meromorphic function,* $L$ *be a non-constant* $L$*-function, and* $n, k$ *be positive integers such that* $n > 3k + 6$. *Suppose that* $(f^n)^{(k)}$ *and* $(L^n)^{(k)}$ *share the value* 1 *CM. Then* $f = tL$, *where* $t$ *is a constant satisfying* $t^n = 1$.

By 2019, X.-M. Li, F. Liu, and H.-X. Yi [43] continued to extend to differential polynomials with a more general structure and obtained the result:

**Theorem 8** ([43]). *Let* $f$ *be a non-constant meromorphic function,*

*$L$ be a non-constant $L$-function, $n, k$ be positive integers, $k \geq 2$ and $n > 3k + 9$. Suppose that the differential polynomials $(f^n(f-1))^{(k)}$ and $(L^n(L-1))^{(k)}$ share the set $\{1\}$ CM. Then $f = L$.*

It can be said that the research problem on uniqueness for meromorphic functions through the condition of sharing one or two sets by meromorphic functions or their differential polynomials has been and is developing strongly, attracting the research interest of many domestic and foreign authors, yielding many important results, while some open problems still need to be further investigated.

Our choice of the topic *On unique polynomials and unique range sets for L-functions and meromorphic functions* is intended to enrich the research results in this direction.

# 2. Objects and objectives of the research

## 2.1. Objects of the research

The research objects of the dissertation are meromorphic functions on the complex plane $\mathbb{C}$ and the class of $L$-functions belonging to the Selberg class $\mathcal{S}$ with strict analytic characterizations.

## 2.2. Objectives of the research

In this dissertation, we focus on solving the following two problems:

**The first problem:** Construct some forms of uniqueness polynomials for meromorphic functions, from which indicate some forms of unique range sets or bi-unique range sets for an $L$-function and a meromorphic function.

**The second problem:** Establish some new uniqueness theorems related to the condition that differential polynomials of an $L$-function and a meromorphic function share a finite set.

## 3. Overview of the dissertation

a) Regarding the first research problem:

Let $P(z)$ be a polynomial in one variable, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two classes of meromorphic functions on $\mathbb{C}$. We say that $P(z)$ is a uniqueness polynomial for $\mathcal{F}_1 \times \mathcal{F}_2$ if the condition $P(f) = P(g)$ implies $f = g$ for all $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_2$. We say that $P(z)$ is a strong uniqueness polynomial for $\mathcal{F}_1 \times \mathcal{F}_2$ if the condition $P(f) = cP(g)$, with $c$ being a non-zero constant, implies $f = g$ for all $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_2$. In the case where $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$, we call $P(z)$ a uniqueness polynomial or a strong uniqueness polynomial for the family $\mathcal{F}$, respectively.

In this dissertation, an important tool that we apply throughout is H. Fujimoto's conditions on strong uniqueness polynomials.

Let $P(z)$ be a polynomial in $\mathbb{C}[z]$ of degree $q$ such that the derivative of $P(z)$ has $k$ distinct zeros $d_1, d_2, \ldots, d_k$ with respective multiplicities $m_1, m_2, \ldots, m_k$.

$$P'(z) = aq(z - d_1)^{m_1}(z - d_2)^{m_2} \cdots (z - d_k)^{m_k}. \qquad (1.3)$$

Then the number $k$ is called the derivative index of $P(z)$. Suppose that:

$$P(d_i) \neq 0, \quad 1 \leq i \leq k. \qquad (1.4)$$

$$m_1 \geq m_2 \geq \cdots \geq m_{k-1} \geq m_k \geq 1, \quad k \geq 2. \qquad (1.5)$$

Note that condition (1.4) implies that $P(z)$ has only simple zeros.

We recall a condition posed by the mathematician H. Fujimoto (see [16], Theorem 1, page 1176):

$$P(d_i) \neq P(d_j), \quad 1 \leq i < j \leq k, \quad k \geq 2. \tag{F}$$

Note that condition (F) is always true when $k = 2$. Therefore, we only need to be concerned with this condition when $k \geqslant 3$.

Denote by $\mathcal{S}$ the Selberg class of $L$-functions, and $\mathcal{M}(\mathbb{C})$ the family of meromorphic functions on $\mathbb{C}$.

For the problem of constructing some forms of uniqueness polynomials for $\mathcal{S} \times \mathcal{M}(\mathbb{C})$, we obtain two results: Theorem 1.2.3 and Theorem 2.2.1 determine the forms of uniqueness polynomials for $\mathcal{S} \times \mathcal{M}(\mathbb{C})$. It can be seen that Theorem 1.2.3 is an extension of Lemma 9 by H. H. Khoai, V. H. An, and N. D. Phuong in [34]. Based on Theorems 1.2.3 and 2.2.1, we prove some results on unique range sets for meromorphic functions as follows:

For the class of $L$-functions and meromorphic functions with finitely many poles, we have established a sufficient condition relating to the degree of the polynomial for an $L$-function and a meromorphic function with finitely many poles to be identical when sharing a finite set (Theorem 1.2.4). From this, we indicate the existence of a class of unique range sets consisting of 5 elements for $L$-functions and meromorphic functions with finitely many poles (Theorem 1.2.4). In addition, we construct new sufficient conditions for a meromorphic function with finitely many poles and an $L$-function to be linearly dependent (Theorem 1.3.1) or identical (Theorem 1.3.2) when they share two classes of distinct finite sets.

It can be seen that Theorem 1.2.4 improves Theorem 2 of H. H. Khoai, V. H. An, and N. D. Phuong in [34] by optimizing the number of elements in the unique range set (from 9 down to 5 elements) and extends Theorem 5 of Q.-Q. Yuan, X.-M. Li, and H.-X. Yi in [70] by removing the hypothesis $\gcd(n, m) = 1$. Theorem 1.3.1 is a generalization of Theorem 5 by Q.-Q. Yuan, X.-M. Li, and H.-X. Yi in [70] and Theorem 2 by Q.-Q. Yuan, X.-M. Li, and H.-X. Yi in [34] by transferring the problem from sharing one set to sharing two distinct finite sets.

For the class of any $L$-functions and meromorphic functions, we have indicated a class of unique range sets for $\mathcal{S} \times \mathcal{M}(\mathbb{C})$ related to the zeros of a polynomial defined by Fujimoto with the minimum number of elements being 5 (Theorem 2.2.2) or related to the zeros of a Yi-type polynomial (Theorem 2.2.7), from which we indicate the existence of a class of Yi-type unique range sets consisting of 10 elements (Remark 2.2.8). Simultaneously, the dissertation also establishes an algebraic condition related to the zeros of two Yi-type polynomials for a meromorphic function to be expressed linearly in terms of an $L$-function or for two $L$-functions to be identical (Theorem 2.2.6). In addition, we establish sufficient conditions relating to the common defect quantity at 0 and $+\infty$ for two meromorphic functions to be identical (Theorem 2.3.1) when they share a finite set. From this, we point out the existence of classes of unique range sets with significantly reduced numbers of elements (down to between 6 and 10 elements) when the meromorphic functions have appropriate deficient values (Corollary 2.3.2 and Corollary 2.3.3).

It can be seen that Theorem 2.2.1 is an extension of Theorem 3.4 by A. Banerjee and A. Kundu in [8] by modifying the hypothesis conditions, allowing the establishment of strong uniqueness polynomial classes on the space $\mathcal{S} \times \mathcal{M}(\mathbb{C})$ for polynomials whose derivative root structure contains simple roots, which is a class of objects outside the scope of Theorem 3.4. Next, Theorem 2.2.2 improves Theorem 1 of H. H. Khoai, V. H. An, and N. D. Phuong in [34] by relaxing the polynomial degree condition. Theorem 2.2.6 is a direct extension of Theorem 1 by V. H. An and N. D. Phuong in [5]. Especially, the result in Theorem 2.3.1 concretizes and fully proves the conjecture of H. H. Khoai and V. H. An in [29] regarding the relationship between meromorphic functions and $L$-functions, through quantifying the conditions on deficient values for the identity of the two function classes.

b) Regarding the second research problem, we have:

Established sufficient conditions related to sharing a finite set by two corresponding differential polynomials of type $(f^n)^{(m)}$ for two meromorphic functions to be linearly expressed in terms of each other or for their higher-order derivatives to be identical (Theorem 3.2.1, Corollary 3.2.3, and Corollary 3.2.4). From this, we established a unique range set for meromorphic functions consisting of 11 elements related to the common defect quantity at 0 and $+\infty$ of two differential polynomials of type $(f^n)^{(m)}$ (Corollary 3.2.2).

Established a sufficient condition related to differential polynomials of the form $(f^n)^{(m)}$ sharing a finite set or an element for an $L$-function and a meromorphic function to be identical or to differ by a proportional constant which is a root of unity (Theorem 3.2.9, Corollary 3.2.10).

Constructed some new uniqueness theorems relating to differential polynomials of the form $(f^n(af^m+b))^{(k)}$ sharing a finite set or an element for an $L$-function and a meromorphic function to be identical or to be linearly expressed in terms of each other (Theorem 3.2.11, Corollary 3.2.12, and Theorem 3.2.13).

It can be seen that Theorem 3.2.9 extended Theorem 1.1 of F. Liu, X.-M. Li, and H.-X. Yi in [48] by establishing the sharing of the set of $d$-th roots of unity CM for the $k$-th order derivative of the monomial $(f^n)^{(k)}$. Furthermore, Theorem 3.2.11 and Corollary 3.2.12 are a direct improvement and generalization of Theorem 1.10 by X.-M. Li, F. Liu, and H.-X. Yi in [43] by relaxing the structure of the investigated object to the derivative of the class of differential polynomials generated by a binomial of the form $(f^n(af^m + b))^{(k)}$ under specific algebraic constraints on the degree $n$. Following that approach, Theorem 3.2.13 further generalized the above results for the derivative of a higher-degree polynomial class of the form $(f^n Q(f))^{(k)}$ (where $Q(z)$ is a polynomial of degree $m$) by establishing the identity condition $f = L$ based on the hypothesis that $z^n Q(z)$ is a uniqueness or strong uniqueness polynomial for meromorphic functions.

# 4. Research methods and tools

The dissertation uses the tools of complex analysis, Nevanlinna value distribution theory, and the theory of $L$-functions in the Selberg class. Regarding methodology, the dissertation utilizes the auxiliary function technique to eliminate the influence of infinitely many poles in proofs for general meromorphic functions. Concurrently, the research applies the technique of comparing and contrasting the inverse image structures of finite sets through strong uniqueness polynomials based on degree conditions.

Regarding tools, the dissertation applies the First and Second Fundamental Theorems of Nevanlinna Theory as the analytical foundation. Specialized characteristics of $L$-functions in the Selberg class $\mathcal{S}$, such as the functional equation, Euler product hypothesis, and analytic continuation, are exploited to establish characteristic estimates. The research process is carried out based on combining Nevanlinna inequalities with the algebraic structure of uniqueness polynomials to solve the uniqueness problem.

Besides being published in journals, the main results of the dissertation have been reported at:

- The Seminar of the Department of Analysis and Applied Mathematics, Faculty of Mathematics, University of Education, Thai Nguyen University.

# Chapter 1

# Uniqueness Problem for *L*-functions and Meromorphic Functions with Finitely Many Poles

## 1.1 Some concepts and auxiliary results

In this chapter, we use the standard notations and definitions of Nevanlinna value distribution theory. Regarding $L$-functions, we recall the following definition:

**Definition 1.1.14** ([60], pp. 111-112)**.** An $L$-function is always understood as an $L$-function in the Selberg class $\mathcal{S}$. An $L$-function in the Selberg class $\mathcal{S}$ is defined as a Dirichlet series absolutely convergent in the half-plane $\mathrm{Re}(z) > 1$:

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},\ a(1) = 1,$$

and satisfies the following conditions:

(i) Ramanujan hypothesis: for every positive number $\epsilon$, $a(n) \ll n^{\epsilon}$;

(ii) Analytic continuation: there exists a non-negative integer $m$ such that $(s-1)^m L(s)$ is an entire function of finite order.

(iii) Functional equation: there exist positive real numbers $Q$, $\lambda_i$, and a positive integer $K$, and complex numbers $\mu_i$, $\omega$ with $\operatorname{Re} \mu_i \geqslant 0$ and $|\omega| = 1$ such that

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1-\bar{s})},$$

where

$$\Lambda_L(s) = L(s) Q^s \prod_{i=1}^{K} \left(\lambda_i s + \mu_i\right).$$

(iv) Euler product hypothesis:

$$L(s) = \prod_p L_p(s),$$

where

$$L_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{b\left(p^k\right)}{p^{ks}}\right)$$

with the coefficients $b\left(p^k\right)$ satisfying $b\left(p^k\right) \ll p^{k\theta}$ for some $\theta \leq \frac{1}{2}$, and the product is taken over all prime numbers $p$.

## 1.2 Unique range sets for $L$-functions sharing a finite set with meromorphic functions having finitely many poles

**Definition 1.2.1.** A polynomial $P(z)$ is called a **uniqueness polynomial on** $\mathbb{A} \times \mathbb{B}$ if for any two non-constant meromorphic functions

$f, g$ such that $(f, g) \in \mathbb{A} \times \mathbb{B}$, the condition $P(f) = P(g)$ implies $f = g$.

**Definition 1.2.2.** A polynomial $P(z)$ is called a **strong uniqueness polynomial on** $\mathbb{A} \times \mathbb{B}$ if for any two non-constant meromorphic functions $(f, g)$ such that $(f, g) \in \mathbb{A} \times \mathbb{B}$, the condition $P(f) = cP(g)$, $c \neq 0$, implies $f = g$.

Let $P(z)$ be a polynomial in $\mathbb{C}[z]$ of degree $q$ of the form:

$$P(z) = a(z - a_1) \cdots (z - a_q), \quad a \neq 0,$$

with the derivative of $P(z)$ having $k$ distinct zeros $d_1, d_2, \ldots, d_k$ with respective multiplicities $m_1, m_2, \ldots, m_k$, and then:

$$P'(z) = aq(z - d_1)^{m_1}(z - d_2)^{m_2} \cdots (z - d_k)^{m_k}. \tag{1.3}$$

The number $k$ is called the derivative index of $P(z)$.

Suppose that:

$$P'(d_i) \neq 0, \quad 1 \leq i \leq k. \tag{1.4}$$

$$m_1 \geq m_2 \geq \cdots \geq m_{k-1} \geq m_k \geq 1, \quad k \geq 2. \tag{1.5}$$

Note that condition (1.4) implies $P(z)$ has only simple zeros. We recall the following condition established by Fujimoto (see [16], Theorem 1, p. 1176):

$$P(d_i) \neq P(d_j), \quad 1 \leq i < j \leq k, \quad k \geq 2. \tag{F}$$

**Theorem 1.2.3.** *Let $P(z)$ be defined as in (1.3) with conditions (1.4), (1.5), and (F). Suppose that one of the following conditions is satisfied:*

*1) If $k \geq 2$, $q \geq 4$ and $m_2 = 1$ or $q \geq 6$ and $m_2 \geq 2$.*

*2) If $k = 3$, $m_1 \geq 2$.*

*3) If $k \geq 4$.*

*Then $P(z)$ is a uniqueness polynomial on $\mathcal{S} \times \mathcal{M}(\mathbb{C})$.*

**Theorem 1.2.4.** *Let $f$ be a non-constant meromorphic function with finitely many poles, let $L$ be an $L$-function, and let $P(z)$ be defined as in (1.3) with conditions (1.4), (1.5), (F), and suppose $P(z)$ is a uniqueness polynomial on $\mathcal{S} \times \mathcal{M}(\mathbb{C})$, $S = \{z : P(z) = 0\}$. Suppose $q \geq 2(m_2 + \cdots + m_k) + 3$. Then if $E_L(S) = E_f(S)$, we have $L = f$.*

From Theorem 1.2.3 and Theorem 1.2.4, we obtain the following corollary. Let $n, m \in \mathbb{N}^*$, $a \in \mathbb{C}$, $a \neq 0$. Consider the polynomial $P_K(z) \in \mathbb{C}[z]$ of the following form:

$$\begin{aligned} P_K(z) &= (n+m+1)\left(\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i} z^{n+m+1-i} a^i\right) + 1 \\ &= Q_K(z) + 1 \end{aligned} \tag{1.13}$$

where:

$$Q_K(z) = (n+m+1)\left(\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i} z^{n+m+1-i} a^i\right)$$

Suppose that:

$$\begin{aligned} Q_K(a) = (n+m+1)\left(\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i}\right) a^{n+m+1} \neq -1, \\ \text{which means } P_K(a) \neq 0, \end{aligned} \tag{1.14}$$

and suppose:

$$a \neq 0, \quad Q_K(a) \neq -1. \tag{1.15}$$

Then $P_K'(z) = (n+m+1)z^n(z-a)^m$ and $P_K'$ has a zero at 0 of multiplicity $n$ and a zero at $a$ of multiplicity $m$. Note that $P_K(z)$ has simple zeros if and only if condition (1.15) holds.

**Corollary 1.2.5.** *Let $f$ be a non-constant meromorphic function with finitely many poles, let $L$ be an $L$-function, and let $P_K(z)$ be defined as in (1.13) with conditions (1.14) and (1.15), $S_K = \{z : P_K(z) = 0\}$. Then:*

*i) If $n \geq 3, m = 1$, then $P_K(z)$ is a uniqueness polynomial on $\mathcal{S} \times \mathcal{M}(\mathbb{C})$. ii) If $E_L(S_K) = E_f(S_K)$ and $n \geq m+2$, then $L = f$.*

**Remark 1.2.6.** i) Corollary 1.2.5, part ii) is precisely Theorem 2 by H. H. Khoai, V. H. An, and N. D. Phuong [34]. Thus, Theorem 1.2.4 is an extension of Theorem 2 [34].

ii) Theorem 5 by Q.-Q. Yuan, X.-M. Li, and H.-X. Yi [70] provides a class of unique range sets of an $L$-function sharing a set with 7 elements with a non-constant meromorphic function having finitely many poles. Corollary 1.2.5, part ii) provides a class of unique range sets of an $L$-function sharing a set with 5 elements with a non-constant meromorphic function having finitely many poles. Thus, Theorem 1.2.4 is an improvement of Theorem 5 [70].

## 1.3 Linear dependence for $L$-functions and meromorphic functions with finitely many poles sharing finite sets

Consider the polynomials $P(z)$ and $Q(z) \in \mathbb{C}[z]$ of degree $n$ having the following forms:

$$P(z) = az^n + bz^{n-m} + c, \quad \text{where } a, b, c \neq 0, \tag{1.16}$$

$$Q(z) = uz^n + vz^{n-m} + t, \quad \text{where } u, v, t \neq 0. \tag{1.17}$$

Suppose:

$$\frac{a^{n-m}c^m}{b^n} \neq \frac{(-1)^n(n-m)^{n-m}m^m}{n^n}, \tag{1.18}$$

$$\frac{u^{n-m}t^m}{v^n} \neq \frac{(-1)^n(n-m)^{n-m}m^m}{n^n}. \tag{1.19}$$

Note that the polynomial $P(z)$ (respectively $Q(z)$) has $n$ distinct simple zeros if and only if condition (1.18) (respectively (1.19)) is satisfied (see [39], Lemma 2.7, p. 151).

**Theorem 1.3.1.** *Let $m, n$ be positive integers such that $n \geq 2m+3$, let $P(z), Q(z)$ be polynomials of forms (1.16) and (1.17) with conditions (1.18) and (1.19), and let $S, T$ be the zero sets of $P(z), Q(z)$, respectively. Suppose $f$ is a non-constant meromorphic function with finitely many poles in the complex plane $\mathbb{C}$ and $L$ is an $L$-function. Then we have:*

*1) $E_L(S) = E_f(T)$ if and only if $f = hL$ and $hS = T$, where $h$ is a non-zero constant satisfying $h^n = \frac{at}{cu}$ and $h^m = \frac{av}{ub}$.*

*2) $E_{L_1}(S) = E_{L_2}(T)$ if and only if $L_1 = L_2$, $\frac{a}{u} = \frac{b}{v} = \frac{c}{t}$, and $S = T$, where $L_1, L_2$ are L-functions.*

**Theorem 1.3.2.** *Let $m, n$ be positive integers such that $\gcd(n, m) = 1$ and $n \geq 2m + 3$, let $P(z)$ be a polynomial of the form (1.16) with condition (1.18), and let $S$ be the zero set of $P(z)$. Suppose $L$ is an L-function and $f$ is a non-constant meromorphic function with finitely many poles satisfying $E_L(S) = E_f(S)$. Then $f = L$.*

# Chapter 2

# Uniqueness Problem for $L$-functions and Meromorphic Functions

## 2.1 Some concepts and auxiliary results

**Definition 2.1.1.** Let $f$ be a non-constant meromorphic function and $k, d$ be positive integers. We denote by $\overline{N}_{(k}(r, f)$ and $\overline{N}_{k)}(r, f)$ the counting functions of poles of $f$ with multiplicity $\geq k$ and $\leq k$ respectively, where each pole is counted only once. We denote by $\overline{N}_{(k}\left(r, \dfrac{1}{f}\right)$ and $\overline{N}_{k)}\left(r, \dfrac{1}{f}\right)$ the counting functions of zeros of $f$ with multiplicity $\geq k$ and $\leq k$ respectively, where each zero is counted only once.

**Lemma 2.1.4.** *Let $f$ be a non-constant meromorphic function. Then*

$$\overline{N}\left(r, \frac{1}{f}\right) - \frac{1}{2}\overline{N}_{1)}\left(r, \frac{1}{f}\right) \leq \frac{1}{2}N\left(r, \frac{1}{f}\right).$$

## 2.2 Value distribution and unique range sets for $L$-functions sharing a finite set with meromorphic functions

**Theorem 2.2.1.** *Let $P(z)$ be a polynomial of degree $q \geq 4$, defined as in (1.3) with conditions (1.4), (1.5), (1.6) in Chapter 1. Suppose $P(z)$ is a uniqueness polynomial on $\mathcal{S} \times \mathcal{M}(\mathbb{C})$ and one of the following conditions is satisfied:*

*1) $P(d_i)P(d_j) \neq P(d_1)^2$ for all $i, j \in \{1, 2, \ldots, k\}$ and $m_1 \geq 2$.*

*2) $m_1 \geq m_2 + 2$.*

*Then $P(z)$ is a strong uniqueness polynomial on $\mathcal{S} \times \mathcal{M}(\mathbb{C})$.*

As an application of Theorem 2.2.1, we obtain the following Theorem 2.2.2.

**Theorem 2.2.2.** *Let $P(z)$ be a polynomial of degree $q \geq 5$, defined as in (1.3) with conditions (1.4), (1.5) and (1.6) established in Chapter 1. Let $P(z)$ be a strong uniqueness polynomial on $\mathcal{S} \times \mathcal{M}(\mathbb{C})$, and $S = \{z : P(z) = 0\}$. Then $S$ is a unique range set on $\mathcal{S} \times \mathcal{M}(\mathbb{C})$ if the following condition is satisfied: $q \geq 2k + 4$ and $m_2 \geq 2$.*

**Corollary 2.2.3.** *Let $f$ be a non-constant meromorphic function, let $L$ be an $L$-function, and let $P_K(z)$ be defined as in (1.13) with conditions (1.14) and (1.15) in Chapter 1, $S_K = \{z : P_K(z) = 0\}$. Then:*

*i) $P_K(z)$ is a strong uniqueness polynomial on $\mathcal{S} \times \mathcal{M}(\mathbb{C})$ if*

$$P_K(0)+P_K(a) \neq 0 \quad \text{and} \quad n \geq 3, m = 1, \quad \text{or} \quad n \geq m+2 \quad \text{and} \quad m \geq 2.$$

*ii) $L = f$ if $E_L(S_K) = E_f(S_K)$ and $n + m \geq 7$, $n \geq m \geq 2$.*

**Remark 2.2.4.** i) Corollary 2.2.3, part i) provides a class of strong uniqueness polynomials on the space $\mathcal{S} \times \mathcal{M}(\mathbb{C})$ with 5 elements. Meanwhile, Corollary 1.1 (p. 126, [31]) provides a class of strong uniqueness polynomials on $\mathcal{M}(\mathbb{C}) \times \mathcal{M}(\mathbb{C})$ (applied to meromorphic functions) with 6 elements $(6 > 5)$. In Theorem 3.4 (p. 502, [8]), A. Banerjee and A. Kundu assumed that $P(z)$ is a strong uniqueness polynomial for meromorphic functions, meaning they still rely on the strong uniqueness polynomial criterion on $\mathcal{M}(\mathbb{C}) \times \mathcal{M}(\mathbb{C})$. Therefore, Theorem 2.2.1 (the foundation of Corollary 2.2.3 i)) is indeed an important improvement and extension in terms of algebraic structure compared to Theorem 3.4 in [8].

ii) Corollary 2.2.3, part ii) with the degree threshold condition $n + m \geq 7$ is a direct improvement result over Theorem 1 in [34] by the group of authors H. H. Khoai, V. H. An, and N. D. Phuong (2023). Thus asserting, Theorem 2.2.2 is a comprehensive improvement and extension of Theorem 1 [34].

**Theorem 2.2.6.** *Let $m, n$ be positive integers such that $n \geq 2m + 6$ and $m \geq 2$, let $P(z), Q(z)$ be polynomials of forms (1.16) and (1.17) with conditions (1.18) and (1.19) given in Chapter 1, and let $S, T$ be two sets of zeros of $P(z), Q(z)$, respectively. Suppose $f$ is a non-constant meromorphic function in the complex plane and $L$ is an $L$-function. Then we have:*

*1) $E_L(S) = E_f(T)$ if and only if $f = hL$ and $hS = T$, where $h$ is a non-zero constant satisfying $h^n = \frac{at}{cu}$ and $h^m = \frac{av}{ub}$.*

*2) $E_{L_1}(S) = E_{L_2}(T)$ if and only if $L_1 = L_2$, $\frac{a}{u} = \frac{b}{v} = \frac{c}{t}$ and $S = T$, where $L_1, L_2$ are $L$-functions.*

**Theorem 2.2.7.** *Let $m, n$ be positive integers such that $\gcd(n, m) = 1$ and $n \geq 2m+3, m \geq 2$, let $P(z)$ be a polynomial of form (1.16) with condition (1.18) in Chapter 1, and let $S$ be the zero set of $P(z)$. Suppose that $E_L(S) = E_f(S)$ for an $L$-function $L$ and a non-constant meromorphic function $f$. Then we have $f = L$.*

**Remark 2.2.8.** Theorem 2.2.7 gives us a unique range set with 10 elements for an $L$-function and a non-constant meromorphic function when they share a Yi-type set. Such a unique range set is different from the unique range set in Theorem 1 [34].

## 2.3 Value distribution and unique range sets for meromorphic functions sharing a finite set

We define the Nevanlinna deficient values as follows:

$$\Theta(\infty, (f_1, \ldots, f_N)) = 1 - \limsup_{r \to +\infty} \frac{\sum_{i=1}^{N} \overline{N}(r, f_i)}{\sum_{i=1}^{N} T(r, f_i)},$$

$$\Theta(a, (f_1, \ldots, f_N)) = 1 - \limsup_{r \to +\infty} \frac{\sum_{i=1}^{N} \overline{N}\left(r, \frac{1}{f_i - a}\right)}{\sum_{i=1}^{N} T(r, f_i)}.$$

Note that $0 \leq \Theta(\infty, (f_1, \ldots, f_N)) + \Theta(a, (f_1, \ldots, f_N)) \leq 2$.

**Theorem 2.3.1.** *Let $S = \{a_1, a_2, \ldots, a_q\} \subset \mathbb{C}$, and let the polynomial $P(z)$ be defined as in equation (1.3) with conditions (1.4) and (1.6) in Chapter 1. Suppose $P(z)$ is a strong uniqueness polynomial for meromorphic functions. If $k \geq 3$, or $k = 2$ and $\min\{m_1, m_2\} \geq 2$, then for two non-constant meromorphic functions $f, g$, the conditions $E_f(S) = E_g(S)$ and $q > 2k + 6 - 4(\Theta(\infty, (f, g)) + \Theta(0, (f, g)))$ imply $f = g$.*

**Corollary 2.3.2.** *Let $f, g$ be two non-constant meromorphic functions and let $P_K$ be the polynomial defined in Corollary 2.2.3 with $S_K$ being the zero set of $P_K$. Suppose $E_f(S_K) = E_g(S_K)$ and one of the following conditions is satisfied:*

*i)* $q \geq 6$, $\Theta(\infty, (f, g)) + \Theta(0, (f, g)) > 1$, *or*

*ii)* $q \geq 7$, $\Theta(\infty, (f, g)) + \Theta(0, (f, g)) > \frac{3}{4}$, *or*

*iii)* $q \geq 8$, $\Theta(\infty, (f, g)) + \Theta(0, (f, g)) > \frac{1}{2}$, *or*

*iv)* $q \geq 9$, $\Theta(\infty, (f, g)) + \Theta(0, (f, g)) > \frac{1}{4}$, *or*

*v)* $q \geq 10$, $\Theta(\infty, (f, g)) + \Theta(0, (f, g)) > 0$.

*Then* $f = g$.

**Corollary 2.3.3.** *Let $f, g$ be two non-constant meromorphic functions with multiplicities of poles and multiplicities of zeros being at least $s, l$ respectively, and let $P_K$ be the polynomial defined in Corollary 2.2.3 with $S_K$ being the zero set of $P_K$. Suppose $E_f(S_K) = E_g(S_K)$ and one of the following conditions is satisfied:*

*i)* $q \geq 6$, $\frac{1}{s} + \frac{1}{l} < 1$, *or*

*ii)* $q \geq 7$, $\frac{1}{s} + \frac{1}{l} < \frac{5}{4}$, *or*

*iii)* $q \geq 8$, $\frac{1}{s} + \frac{1}{l} < \frac{3}{2}$, *or*

*iv)* $q \geq 9$, $\frac{1}{s} + \frac{1}{l} < \frac{7}{4}$, *or*

*v)* $q \geq 10$, $\frac{1}{s} + \frac{1}{l} < 2$.

*Then* $f = g$.

# Chapter 3

# Uniqueness Problem for $L$-functions and Meromorphic Functions with Differential Polynomial Conditions

## 3.1 Some concepts and auxiliary results

**Lemma 3.1.5.** *Let $f$ be a non-constant meromorphic function and $n, m$ be two positive integers, $n > m$ and $a$ be a pole of $f$. Then*

$$(f^n)^{(m)} = \frac{\varphi_m}{(z-a)^{np+m}}, \quad \text{where } p = \nu_f^\infty(a), \quad \varphi_m(a) \neq 0.$$

**Lemma 3.1.6.** *Let $f$ be a non-constant meromorphic function and $n, m$ be two positive integers, $n > m$ and $a, b$ be a pole and a zero of $f$, respectively. Then i)* $\dfrac{(f^n)^{(m)}}{f^n} = \dfrac{h_m}{(z-a)^m}$, *where* $p = \nu_f^\infty(a), \quad h_m(a) \neq 0$; *ii)* $\dfrac{(f^n)^{(m)}}{f^n} = \dfrac{S_m}{(z-b)^m}$, *where* $l = \nu_f^0(b), \quad S_m(b) \neq 0.$

**Lemma 3.1.7.** *Let $f$ be a non-constant meromorphic function, $n, m, k$ be integers, $m \geq 0$, $n > k+1$, and $P(z)$ be a polynomial of degree $m$. Then*

$$\begin{aligned}(n-k-1)T(r,f) &\leq T\left(r, (f^nP(f))^{(k)}\right) + S(r,f)\\ &\leq (n+m)(k+1)T(r,f) + S(r,f).\end{aligned}$$

*In particular, $(f^nP(f))^{(k)}$ is non-constant and $S\left(r, (f^nP(f))^{(k)}\right) = S(r,f)$.*

**Lemma 3.1.8.** *Let $f$ be a non-constant meromorphic function, $n, m, k, d$ be integers, $m \geq 0$, $k > 0$, $n > k + 1 + \frac{2k+m+2}{d}$ and $P(z)$ be a polynomial of degree $m$, $S = \{a \in \mathbb{C} : a^d = 1\}$. Then*

$$(d(n-k-1)-2k-m-2)T(r,f) \leq \overline{N}\left(r, \frac{1}{((f^nP(f))^{(k)})^d - 1}\right) + S(r,f).$$

*In particular, $\left((f^nP(f))^{(k)}\right)^{-1}(S) \neq \emptyset$.*

## 3.2 Value distribution and unique range sets for differential polynomials of $L$-functions and meromorphic functions sharing the set of roots of unity

**Theorem 3.2.1.** *Let $f$ and $g$ be two non-constant meromorphic functions and $n, m$ be positive integers, let $P(z)$ be a polynomial whose derivative with index $k$ satisfies Fujimoto conditions (F). Suppose $P(z)$ is a strong uniqueness polynomial for meromorphic functions. If $k \geq 3$, or $k = 2$ and $\min\{m_1, m_2\} \geq 2$, then we have:*

*1) If $E_{(f^n)^{(m)}}(S) = E_{(g^n)^{(m)}}(S)$, $n \geq m+2$, and $q > 2k+6-\frac{8(n-1)}{n+m}$, then $f = cg$ with $c^n = 1$.*

*2) If $E_{f^{(m)}}(S) = E_{g^{(m)}}(S)$, and $q > 2k+6-\frac{4m}{m+1}$, then $f^{(m)} = g^{(m)}$.*

**Corollary 3.2.2.** *Let $f$ and $g$ be two non-constant meromorphic functions and let $P_K(z)$ be the polynomial defined as above with $S_K$ being the zero set of $P_K$, respectively. If*

$$k = 2, \quad \Theta(\infty, ((f^n)^{(m)}, (g^n)^{(m)})) = 0, \quad \Theta(0, ((f^n)^{(m)}, (g^n)^{(m)})) = 0,$$

*then $S_K$ is a unique range set for meromorphic functions with* 11 *being its minimum number of elements.*

**Corollary 3.2.3.** *Let $f, g$ be two non-constant meromorphic functions, $n, m$ be positive integers, and let $P_K(z)$ be the polynomial as above with $S_K$ being the zero set of $P_K$. Suppose $E_{(f^n)^{(m)}}(S_K) = E_{(g^n)^{(m)}}(S_K)$ and one of the following conditions is satisfied:*

*i) $q \geq 6$ and $n \geq m+3$, or*

*ii) $q \geq 7$ and $n \geq m+2$.*

*Then $f = cg$ with $c^n = 1$.*

**Corollary 3.2.4.** *Let $f, g$ be two non-constant meromorphic functions, $n, m$ be positive integers, and let $P_K(z)$ be the polynomial defined as above with $S_K$ being the zero set of $P_K$. Suppose $E_{f^{(m)}}(S_K) = E_{g^{(m)}}(S_K)$ and one of the following conditions is satisfied:*

*i) $q \geq 7$ and $m > 3$.*

*ii) $q \geq 8$ and $m > 1$.*

*iii) $q \geq 9$.*

*Then $f^{(m)} = g^{(m)}$.*

**Lemma 3.2.5.** *Let $f$ be a non-constant meromorphic function, $L$ be an $L$-function, $n, m, k$ be integers, $m \geq 0$, $k > 0$, and let $P, Q$ be polynomials of degree $m$. Suppose $(f^nP(f))^{(k)}$ and $(L^nQ(L))^{(k)}$ share 1 CM, and $n > 3k+m+3$. Then, we have one of the following cases:*

*(1)* *(i)* $(n-k-2)T(r,L)+S(r,L) \leq (m+2k+4)T(r,f)+S(r,f)$,

*(ii)* $(n-k-4)T(r,f)+S(r,f) \leq (m+k+2)T(r,L)+S(r,L)$,

*(iii)* $S(r,f) = o(T(r,L))$, $S(r,L) = o(T(r,f))$.

*(2)* $(L^nQ(L))^{(k)}(f^nP(f))^{(k)} = 1$.

*(3)* $(L^nQ(L))^{(k)} = (f^nP(f))^{(k)}$.

**Lemma 3.2.6.** *Let $f$ be a non-constant meromorphic function, $L$ be an $L$-function, $n, m, k, d$ be integers, $m \geq 0, k > 0$, let $P(z)$ and $Q(z)$ be polynomials of degree $m$, and let $S = \{a \in \mathbb{C} : a^d = 1\}, d \geq 2$. Suppose $(f^nP(f))^{(k)}$ and $(L^nQ(L))^{(k)}$ share $S$ CM and*

$$n > k+1+\frac{2k+m+2}{d}.$$

*Then, we have one of the following three cases:*

*(1)* *(i)* $d(n-k-1)T(r,L)+S(r,L) \leq 2(m+2k+2)T(r,f)+S(r,f)$,

*(ii)* $(d(n-k-1)-2)T(r,f)+S(r,f) \leq 2(m+k+1)T(r,L)+S(r,L)$,

*(iii)* $S(r,f) = o(T(r,L))$, $S(r,L) = o(T(r,f))$.

*(2)* $(L^nQ(L))^{(k)}(f^nP(f))^{(k)} = h, h^d = 1$.

*(3)* $(L^nQ(L))^{(k)} = h(f^nP(f))^{(k)}, h^d = 1$.

**Lemma 3.2.7.** *Let $n, m, k$ be integers, $m \geq 0$, $n > m+2$, $k > 0$, and let $P(z)$ and $Q(z)$ be polynomials of degree $m$. If for a non-constant meromorphic function $f$ and an $L$-function $L$, $(L^n Q(L))^{(k)} = h(f^n P(f))^{(k)}$, where $h^d = 1$, then $L^n Q(L) = h f^n P(f)$.*

**Lemma 3.2.8.** *Let $a, b, c, u$ be non-zero complex constants, $n$ and $m$ be positive integers, $n \geq m+3$. Set*

$$R(z) = az^n + bz^{n-m}, \quad T(z) = uz^n + vz^{n-m}.$$

*i) If $R(L) = T(f)$ for $L \in S$ and $f \in \mathcal{M}(\mathbb{C})$, then $f = hL$, where $h$ is a constant, satisfying $h^n = \frac{a}{u}, h^{n-m} = \frac{b}{v}$.*

*ii) If $\gcd(n, m) = 1, c = a, u = b$, and $R(L) = T(f)$ for $L \in S$ and $f \in \mathcal{M}(\mathbb{C})$, then $f = L$.*

*iii) If $R(L_1) = T(L_2)$ for $L_j \in S\,(j = 1, 2)$, then $L_1 = L_2$ and $a = u, b = v$.*

**Theorem 3.2.9.** *Let $f$ be a non-constant meromorphic function, $L$ be a non-constant $L$-function, $n, k$ be positive integers, and let $S$ be the set of $d$-th roots of unity. Suppose that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $S$ CM. Then $f = tL$, where $t^{nd} = 1$, if one of the following two conditions is satisfied:*

*1) $d = 1$ and $n > 3k + 6$;*

*2) $d \geq 2$, and $n > k + 1 + \frac{4k+4}{d}$.*

**Corollary 3.2.10.** *Let $f$ be a non-constant meromorphic function, $L$ be a non-constant $L$-function, $n, k$ be positive integers. Suppose that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share 1 CM. Then $f = tL$, where $t^n = 1$, if $d = 1$ and $n > 3k + 6$.*

**Theorem 3.2.11.** *Let $f$ be a non-constant meromorphic function, $L$ be a non-constant L-function, $n, m, k$ be positive integers, $k \geq 2, a, b, c, u \in \mathbb{C}, a, b, c, u \neq 0$, and let $S$ be the set of $d$-th roots of unity. From the hypothesis that two functions $(L^n(cL^m + u))^{(k)}$ and $(f^n(af^m + b))^{(k)}$ share the set $S$ CM, we obtain $f = tL$ for a constant $t$ such that $t^{(n+m)d} = \frac{c^d}{a^d}$ and $t^{nd} = \frac{u^d}{b^d}$, if one of the following two conditions is satisfied:*

1) $d = 1, n > 3k + m + 6$;
2) $d \geq 2, n > \max\left\{k + 1 + \dfrac{4k + 2m + 4}{d}, m + 2\right\}$.

*In a particular case, if $a = c$, $b = u$, $\gcd(n, m) = 1$ and $d = 1$, then $f = L$.*

**Corollary 3.2.12.** *Let $f$ be a non-constant meromorphic function, $L$ be a non-constant L-function, $n, k$ be positive integers, $n > 3k+9, k \geq 2$. If $(f^n(f-1))^{(k)}$ and $(L^n(L-1))^{(k)}$ share 1 CM, then $f = L$.*

**Theorem 3.2.13.** *Let $f$ be a non-constant meromorphic function, $L$ be a non-constant L-function, $n, m, k$ be positive integers, $k \geq 2$, let $S$ be the set of $d$-th roots of unity, and let $Q(z)$ be a polynomial of degree $m$. Suppose that $(f^nQ(f))^{(k)}$ and $(L^nQ(L))^{(k)}$ share $S$ CM. Then $f = L$ if one of the following two conditions is satisfied:*

1) $d = 1, n > 3k + m + 6$ *and $z^nQ(z)$ is a uniqueness polynomial for meromorphic functions;*
2) $d \geq 2, n > \max\left\{k + 1 + \frac{4k+2m+4}{d}, m + 2\right\}$, *and $z^nQ(z)$ is a strong uniqueness polynomial for meromorphic functions.*

# General Conclusion

In this dissertation, we focus on studying the uniqueness problem for $L$-functions in the Selberg class $\mathcal{S}$ and meromorphic functions on the complex plane $\mathbb{C}$ through sharing finite sets and differential polynomials.

**The main results of the dissertation include:**

1. Studying the uniqueness problem for $L$-functions and meromorphic functions with finitely many poles by establishing a sufficient condition for a polynomial to be a uniqueness polynomial (**Theorem 1.2.3**). From this, successfully constructing classes of finite unique range sets $S$ with exactly 5 elements for the pair of functions $(L, f)$ (**Theorem 1.2.4**, **Corollary 1.2.5**, and **Theorem 1.3.2**).

2. Extending the uniqueness problem to the general case and optimizing the cardinality of unique range sets based on analytic characteristics. Specifically, establishing uniqueness theorems when sharing one or two finite sets (**Theorem 2.2.1** and **Theorem 2.2.6**) and proving that the cardinality of the finite unique range set $S$ can be significantly reduced when the meromorphic functions have Nevanlinna deficient values (**Theorem 2.3.1**, **Corollary 2.3.2**, and

**Corollary 2.3.3**).

3. Solving the uniqueness problem for higher-order differential operators. The dissertation establishes uniqueness for higher-order derivatives (**Theorem 3.2.1** and **Theorem 3.2.9**) along with complex differential polynomial structures of the form $(f^n(af^m+b))^{(k)}$ and $(f^nQ(f))^{(k)}$ (**Theorem 3.2.11** and **Theorem 3.2.13**) when they share the set of roots of unity CM with an $L$-function.

**We propose some directions for further research:**

1. Investigating some new forms of uniqueness theorems for $L$-functions and meromorphic functions.

2. Continuing to study the uniqueness problem for $L$-functions and holomorphic curves.

# List of Author's Publications

1. [3] V. H. An, N. X. Lai, N. D. Phuong (2021), Uniqueness of Meromorphic Functions with Deficient values and unique range sets of small cardinalities, *Ann. Univ. Sci. Budapest. Sect. Comp.*, 52, pp. 13–28.

2. [34] H. H. Khoai, V. H. An, N. D. Phuong (2023), On value distribution of $L$-functions sharing finite sets with meromorphic functions, *Bull. Math. Soc. Sci. Math. Roumanie*, 66(114), No. 3, pp. 265–280.

3. [4] V. H. An, H. H. Khoai, N. D. Phuong (2024), On value sharing of certain differential polynomials of meromorphic functions and $L$-functions, *The Journal of Analysis*, 32, pp. 2485–2502.

4. [5] V. H. An, N. D. Phuong (2025), The linear dependency of $L$-functions and meromorphic functions sharing finite sets, *J. Math. Math. Sci.*, pp. 57–72.