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ON THE CARTAN-TYPE SECOND MAIN THEOREM FOR REDUCED COUNTING FUNCTION AND UNIQUENESS PROBLEM

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Introduction

1. History and reasons for choosing the topic

In recent years, Value Distribution Theory for holomorphic curves, also known as Nevanlinna-Cartan Theory, has attracted the research attention of many domestic and foreign mathematicians. Considered to have begun with the works of H. Cartan in 1933 when he formulated the first and second forms of fundamental theorems for holomorphic curves, the Nevanlinna-Cartan Theory is considered one of its achievements. The results are profound, beautiful and have many applications in different fields of Mathematics such as the unique problem for holomorphic curves, degeneracy of algebraic curves, dynamical system theory, differential equations. complexes and a number of other fields.

Let $\mathcal{K}$ be an algebraically closed field of characteristis 0, completed with respect to a non-Archimedean absolute value, $\mathbb{W}$ is $\mathbb{C}$ or $\mathcal{K}$ and $\mathbb{P}^n(\mathbb{W})$ be the projective space of dimension n over $\mathbb{W}$. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map where $(f_0, \ldots, f_n)$ be a reduced representative of f. The function

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta$$

is called *the Nevanlinna-Cartan characteristic function* of f, where $\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$.

Let H be a hyperplane, L be a linear form defined in H and let M is a positive integer. Let $n_f(r, H)$ be the number of zeros of $L(f)(z)$ in the disk $\{|z| \leqslant r\}$, counting multiplicity, and $n_f^M(r, H)$ be the number of zeros of $L(f)(z)$ in the disk $\{|z| \leqslant r\}$, truncated multiplicity by a positive integer M. The function

$$N_f(r, H) = N_f(r, L) = \int_0^r \frac{n_f(t, H) - n_f(0, H)}{t} dt + n_f(0, H) \log r$$

is called *the counting function* and the function

$$N_f^M(r,H) = N_f^M(r,L) = \int_0^r \frac{n_f^M(t,H) - n_f^M(0,H)}{t} dt + n_f^M(0,H)\log r$$

is called *the truncated counting function* by M of f with respect to H.

In 1933, H. Cartan proved the following:

Theorem A. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in the general position. Thus, we get*

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} N_f^n(r, H_j) + o(T_f(r))$$

holds for $r > 0$ outside a set of finite Lebesgue measure.

Theorem A gives us a relationship between the characteristic function of linear non-degenerate holomorphic curves with truncated multiple count functions with the target being hyperplanes in general position. This work of H. Cartan is considered extremely important, opening up a new research direction for developing Nevanlinna theory - researching the second main theorem forms for meromorphic, holomorphic mapping and other forms of theorem. Recent research results in this direction is focus on two issues:

1. Construct the forms of Second Main Theorem for holomorphic curves from $\mathbb{W}^p$ or a domain in $\mathbb{W}^p$ to $\mathbb{P}^n(\mathbb{W})$ or a algebraic variety map in $\mathbb{P}^n(\mathbb{W})$ with the target of the hyperplanes, fixed hypersurfaces or moving by establishing a relationship between the Nevanlinna-Cartan characteristic with the proximity functions or different form of counting functions.

2. Research the applications of forms of the Second Main Theorem in various areas of mathematics, for example, the properties of defect, the unique problem for holomorphic functions or curves, the degeneracy of algebraic curves and some other fields.

According to the first research direction, following the work of H. Cartan, many authors have built forms of the second main theorem by establishing inequality relationships between the Characteristic function of a holomorphic curve with proximity and counting functions ignoring of multiples or truncated multiple counting functions. Specifically, in 1983, E. I. Nochka proved an extension of Theorem A to the case of a family of hyperplanes at subgeneral positions in the complex projective space $\mathbb{P}^n(\mathbb{C})$. In 1995, H. H. Khoai and M. V. Tu studied Theorem A for the case of holomorphic curves on the field $\mathcal{K}$ and obtained the results:

Theorem B. *Let $f : \mathcal{K} \to \mathbb{P}^n(\mathcal{K})$ be a linearly non-degenerate holomorphic curve and let $H_1, \ldots, H_q$ be arbitrary hyperplanes in $\mathbb{P}^n(\mathcal{K})$ in general position. Then*

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} N_f^n(r, H_j) - \frac{n(n+1)}{2}\log r + O(1).$$

P. C. Hu and C. C. Yang extended the results of H. H. Khoai and M. V. Tu to the case of the hyperplane family in the subgeneral position. In recent years, many domestic and foreign authors have researched forms of the Second Main Theorem for holomorphic curves from $\mathbb{W}^m$ or a domain over $\mathbb{W}^m$ into $\mathbb{P}^n(\mathbb{W})$ or an internal projective algebraic manifold in $\mathbb{W}$, in cases where the target is fixed or mobile hyperplanes or hypersurfaces, with different types of counting functions. Such as M. Ru, Q. M. Yan and Z. H. Chen, G. Dethloff, T. V. Tan, D. D. Thai, D. D. Thai, S. D. Quang, L. Shi, T. T. H. An and H. T. Phuong, H. T. Phuong and N. V. Thin, H. T. Phuong and L. Vilaisavanh, P. C. Hu, N. V. Thin and many other authors.

In 2014, based on studies on non-point multiplets of non-trivial linear combinations of a finite family of holomorphic functions on the complex plane at a point, J. M. Anderson and A. Hinkkanen introduced a concept new counting function, called the reduced counting function, and prove a form of the Second Main Theorem with this reduced counting function for the case where the target is fixed hyperplanes.

Let $f : \mathbb{W} \to \mathbb{P}^n(\mathbb{W})$ be a holomorphic curve and $(f_0, \ldots, f_n)$ be a reduced representative of f. Denote $\mathcal{L} = \mathcal{L}(f_0, \ldots, f_n)$ be the set of all of non-trivial linear combinations of $f_0, \ldots, f_n$. From Lemma 1.4 we see, for each $z_0 \in \mathbb{W}$, the possible orders of zeros of functons in $\mathcal{L}$ at z_0 form the sequence such that

$$0 = d_0(z_0) < d_1(z_0) < \cdots < d_n(z_0).$$

With the hyperplane H defined by the linear form L, obviously $L(f) \in \mathcal{L}$ should exist $j \in \{0, \ldots, n\}$ such that order of zero of $L(f)$ at z_0 equals $d_j(z_0)$, i.e. $\mathrm{ord}_{L(f)}(z_0) = d_j(z_0)$. We say $\nu(H, z_0) = j$ is the *reduced multiplicity* of zero of $L(f)$ at z_0, also known as the reduced multiple of f combine with the hyperplane H at z_0 and $\varepsilon(H, z_0) = d_j - j$ is the *excess* of $L(f)$ at z_0, also known as the excess multiple of f combine with the hyperplane H at z_0.

Definition 1. For each $r > 0$, we denote $\nu_f(r, H) = \sum\limits_{|z| \leqslant r} \nu(H, z)$. And

$$\mathcal{N}_f(r, H) = \int_0^r \frac{\nu_f(t, H) - \nu_f(0, H)}{t} dt + \nu_f(0, H) \log r$$

function is called *reduced counting function* of the function f combined with the hyperplane H.

From the definition we see that $\nu(H, z_0) \leqslant \min\{d_j, n\}$ so $\nu_f(r, H) \leqslant n_f^n(r, H)$.

$$\mathcal{N}_f(r, H) \leqslant N_f^n(r, H). \tag{1}$$

Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a collection of hyperplanes in general position in $\mathbb{P}^n(\mathbb{W})$ and L_j is a linear form defined as H_j, $j = 1, 2, \ldots, q$. Put

$$H = \frac{L_1(f) L_2(f) \ldots L_q(f)}{W},$$

where W be the Wronskian determinant of $f_0, \ldots, f_n$. From Lemma 1.6, for each arbitrary z_0 we always have $\sum_{j=1}^q \varepsilon(H_j, z_0) \leqslant \mathrm{ord}_W(z_0)$. Put

$$\mathcal{V}(\mathcal{H}, z) = \mathrm{ord}_W(z_0) - \sum_{j=1}^q \varepsilon(H_j, z) \geqslant 0.$$

And denote

$$\mathcal{V}_f(r, \mathcal{H}) = \sum_{|z| \leqslant r} \mathcal{V}(\mathcal{H}, z).$$

Definition 2. Function

$$\mathcal{U}_f(r, \mathcal{H}) = \int_0^r \frac{\mathcal{V}_f(t, \mathcal{H}) - \mathcal{V}_f(0, \mathcal{H})}{t} dt - \mathcal{V}_f(0, \mathcal{H}) \log r$$

is called *the excess multiple count function* at the zero points of the function f combined with the collection of hyperplanes $\mathcal{H}$.

In 2014, J. M. Anderson and A. Hinkkanen proved

Theorem C. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve, and let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a collection of $q \geqslant n+1$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then we have*

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} \mathcal{N}_f(r, H_j) - \mathcal{U}_f(r, \mathcal{H}) - N(r, H) \\ + O(\log r) + O(\log T_f(f)), \tag{2}$$

as $r \to \infty$ outside a set of finite linear measure.

Note that, from (1) we see that the truncated multiple counting function $\mathcal{N}_f(r, H_j)$ in Theorem C is smaller than the truncated multiple counting function in Cartan's work, so it is a resulting improvement of H. Cartan. This work will suggest to us a new research problem in Nevanlinna-Cartan theory: considering the forms of the Second Main Theorem with reduced counting functions.

According to the second research direction, in this thesis we focus on studying the application of Nevanlinna-Cartan theory in the unique problem of meromorphic and holomorphic curves. The first results in this direction of research belong to H. Fujimoto when he extended the Five Point Theorem of R. Nevanlinna to meromorphic mapping. After that, this problem immediately attracted the attention of many authors and obtained many important results.

Let U be a domain in $\mathbb{W}$, denoted by $\mathcal{F}$ as a family of non-constant holomorphic mappings from U into $\mathbb{P}^n(\mathbb{W})$. A collection hypersurfaces $\mathcal{D}$ is said to be *unique range set ignoring multiplicity*, denoted by URSIM (or *unique range set counting multiplicity*, denoted by URSCM) for a family of holomorphic maps $\mathcal{F}$ if for any pair of holomorphic maps $f, g \in \mathcal{F}$, the condition $\overline{E}_f(\mathcal{D}) = \overline{E}_g(\mathcal{D})$ (reps. $E_f(\mathcal{D}) = E_g(\mathcal{D})$) implies $f \equiv g$. The URSIM, URSCM are called *the unique range set* for a family of holomorphic maps $\mathcal{F}$ to the same.

In his works H. Fujimoto proved the existence of unique definite sets including multiples consisting of $3n+2$ hyperplane in general position for the family of linearly non-degenerate complex meromorphic maps and $2n+3$ hyperplane in general position gives a family of complex meromorphic maps that are not algebraically degenerate. In 1983, L. Smiley showed existence of uniquely defined sets regardless of multiples consisting of $3n+2$ hyperplane for complex holomorphic curves and H. Fujimoto studied this problem again in 1998. In 2006, G. Dethloff and T. V. Tan considers the same problem for the case of mobile hyperplanes. In 2006, D. D. Thai and S. D. Quang considered the problem solely in the case where the target is $3n+1$ hyperplane. In 2008, M. Dulock and M. Ru and in 2009, H. T. Phuong proved some results on the unique problem in the case of hypersurfaces for holomorphic curves on complex planes. In 2009, Z. Chen, Q. Yan and in 2010, G. Dethloff, T. V. Tan showed unique sets for holomorphic curves consisting of $2n+3$ hyperplane.

Denote $R_0 > 1$ be a fixed positive real number or $+\infty$ and

$$\Delta = \{z \in \mathbb{C} : \frac{1}{R_0} < |z| < R_0\}$$

be an annulus in $\mathbb{C}$. In 2013, H. T. Phuong and T. H. Minh proved a unique theorem for holomorphic curves on annulus in the case where the target is hyperplanes in general position, and in 2021, H. T. Phuong and L. Vilaisavanh study this problem in the case of hypersurfaces in general positions for Veronese embeddings. Recently, authors have continued to develop unique definitions for different classes of curves with the target being hyperplanes or hypersurfaces, fixed or mobile. Note that most proofs of unique domain results rely on forms of the second main theorem.

Thus, continuing to develop the forms of the second main theorem by establishing the relationship between characteristic functions and counting function forms and the application of these theorems in the unique problem of differential and rectilinear mappings. Image is absolutely necessary. Currently, these research issues are being developed strongly, attracting the attention of many authors and many works being published. The choice of the topic **"On the Cartan-type second main theorem for reduced counting functions and uniqueness problem"** of the author of this thesis is also to further develop some forms of the Fundamental Theorem. secondly with the truncated counting function, the truncated multiple counting function for the holomorphic curve on the field $\mathbb{W}$ and construct some sufficient conditions for the unique problem for the holomorphic curve on an annulus.

2. Purpose and research object

- **Research object:**

  In this thesis, we focus on researching the properties of holomorphic curves on the non-Archimedean field or on the field of complex numbers $\mathbb{C}$ and holomorphic curves on the annulus in the complex plane. These are also the basic research objects of the Nevanlinna-Cartan value distribution theory.

- **Purpose:**

  In this thesis, we research in two directions as follows:

  The first direction of research: Construct some forms of the second main theorem for holomorphic curves on a non-Archimedean field or in the complex plane $\mathbb{C}$ with the targets being hyperplanes in general or sub-general positions by setting The relationship between the Nevanlinna-Cartan characteristic function and the reduced counting function.

  The second direction of research: Establish some sufficient conditions so that the two holomorphic curves on an annulus are coincident in case the target is hypersurfaces in general position.

3. Overview of the thesis

For the first research direction, we have built some forms of the Second Main Theorem as follows:

Theorem 1 (Theorem 1.7, Chapter 1). *Let $f = (f_0 : \cdots : f_n) : \mathcal{K} \to \mathbb{P}^n(\mathcal{K})$ be a linearly non-degenerate holomorphic curve, and let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a collection of $q \geqslant n+1$ hyperplanes in $\mathbb{P}^n(\mathcal{K})$ in general position. Then we have*

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} \mathcal{N}_f(r, H_j) - \mathcal{U}_f(r, \mathcal{H}) - N(r, H) \\ - \frac{n(n+1)}{2}\log r + O(1),$$

as $r \to \infty$ outside a set of finite linear measure.

Theorem 2 (Theorem 2.4, Chapter 2).

Let $f : \mathcal{K} \to \mathbb{P}^n(\mathcal{K})$ be a linearly non-degenerate holomorphic curve, and let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a collection of $q \geqslant 2N - n + 1$ hyperplanes in $\mathbb{P}^n(\mathcal{K})$ in $N-$subgeneral position. Then we have

$$\begin{aligned}(q - 2N + n - 1)T_f(r) \leqslant \sum_{j=1}^{q} \mathcal{N}_f(r, H_j) - \frac{N}{n}\mathcal{U}_f(r, \mathcal{H}) - \frac{N}{Mn}N(r, \Phi)\\ - \frac{(N+1)n}{2}\log r + O(1),\end{aligned}$$

as $r \to \infty$ outside a set of finite linear measure.

Theorems 1 and 2 are two forms of the Second Main Theorem with a reduced counting function for holomorphic curve on non-Archimedean field $\mathcal{K}$ in two cases: a collection of hyperplanes in general position and in subgeneral position in $\mathbb{P}^n(\mathcal{K})$. For the holomorphic curve on the field of complex numbers we obtain the result:

To prove the result on the unique problem for holomorphic curves on an annulus, we proved a form of the Second Main Theorem with truncated multiplets. Specifically, let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$. We denote

$$O_f(r) = \begin{cases} O(\log r + \log T_f(r)) & \text{if} \quad R = +\infty \\ O(\log \dfrac{1}{R_0 - r} + \log T_f(r)) & \text{if} \quad R < +\infty. \end{cases}$$

Theorem 3 (Theorem 2.13, Chapter 2). *Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic curve, and let $D_j, 1 \leqslant j \leqslant q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Let d is the least common multiple of the d_j and set $M = \binom{n+d}{n} - 1$. Then, for any $1 < r < R_0$ and $q \geqslant M + 1$, we have*

$$\| \quad (q - M - 1)T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_f^M(r, D_j) + O_f(r). \tag{3}$$

Theorem 3 is a form of the Second Main Theorem with a truncated multiplicity function for the holomorphic curve on an annulus.

Let $\mathcal{D} = \{D_j, j = 1, \ldots, q\}$ be a collection of q hypersurfaces of degree d_j in general position. Let d be the least common multiple of the d_j. Set

$$\delta_{\mathcal{D}} := \min\{d_1, \dots, d_q\}, \quad n_{\mathcal{D}} = \binom{n+d}{n} - 1$$

and

$$B(\mathcal{D}) = (d(n+1)^2 2^{n+1} + 1)^n.$$

For holomorphic mapping $f : \Delta \to \mathbb{P}^n(\mathbb{C})$, we denote

$$\overline{E}_f(D_j) = \{z \in \Delta \mid (D_j, f)(z) = 0 \text{ ignoring multiplicity}\}$$

and set

$$\overline{E}_f(\mathcal{D}) = \bigcup_{D_j \in \mathcal{D}} \overline{E}_f(D_j).$$

The results on the uniqueness problem for holomorphic curves on an annulus we obtained in this thesis are as follows:

Theorem 4 (Theorem 3.2, Chapter 3). *Let $\mathcal{D} = \{D_1, \dots, D_q\}$ be a collection of q hypersurfaces in general position and $f, g : \Delta \to \mathbb{P}^n(\mathbb{C})$ be algebraically non-degenerate holomorphic curves such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Assume that*

a) $\overline{E}_f(D_i) \cap \overline{E}_f(D_j) = \emptyset$ *for any* $i \neq j \in \{1, \dots, q\}$;

b) $\overline{E}_f(D_j) \subset \overline{E}_g(D_j)$ *for any* $j = 1, 2, \dots, q$ *and* $f(z) = g(z)$ *for all* $z \in \overline{E}_f(\mathcal{D})$.

c) $\liminf\limits_{r \to R_0} \sum_{j=1}^q N_f^1(r, D_j) / \sum_{j=1}^q N_g^1(r, D_j) > \dfrac{M}{M+1}$.

If $q \geqslant 2M + 3$, then there exists a subset $S \subset \{1, \dots, q\}$ such that $\#S > M + 1$ and

$$\frac{(f, D_k)^{d/d_k}}{(f, D_l)^{d/d_l}} \equiv \frac{(g, D_k)^{d/d_k}}{(g, D_l)^{d/d_l}} \quad \text{for all } k \neq l \in S. \tag{4}$$

Theorem 4 gives us a result on the uniqueness problem of algebraically non-degenerate holomorphic curves on annulus sharing hypersurfaces.

Theorem 5 (Theorem 3.5, Chapter 3).

Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \dots, D_q\}$ be a collection of $q > n + 1 + 2Mn/\delta_{\mathcal{D}}$ hypersurfaces in general position $\mathbb{P}^n(\mathbb{C})$ such that $f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$. Then $f \equiv g$.

Theorem 6 (Theorem 3.6, Chapter 3).

Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q > n + 1 + 2M/\delta_{\mathcal{D}}$ hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$ such that

(a) *$f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$,*

(b) *$\overline{E}_f(D_i) \cap \overline{E}_f(D_j) = \emptyset$ and $\overline{E}_g(D_i) \cap \overline{E}_g(D_j) = \emptyset$ for all $i \neq j \in \{1, \ldots, q\}$.*

Then $f \equiv g$.

Theorems 5 and 6 are algebraic conditions to uniquely determine the holomorphic curve on an annulus with the goal of being a collection of hypersurfaces in general position.

4. Research Methods of the thesis

In this thesis, we use basic research methods: based on research-oriented documents, we discover open problems that need to be solved and use the knowledge and techniques of Complex analysis, Nevanlinna-Cartan value distribution theory, linear algebra, algebraic geometry to propose appropriate methods or use some existing techniques to solve the problems.

In addition to publication in journals, the main results of the thesis have been reported at:

- Seminar of the Department of Calculus, Faculty of Mathematics, University of Education, Thai Nguyen University every year.
- International Conference on Algebra - Number Theory - Geometry - Topology 2019, 04 - 08/12/2019 at Ba Ria-Vung Tau Teacher Training college.
- International Conference on Algebra - Number Theory - Geometry - Topology 2021, 21 - 23/10/2021 at University of Education, Thai Nguyen University.

Chapter 1
Second main theorem with reduced counting function for holomorphic curves on the non-Archimedean field

1.1. Some basic knowledge

Let $\mathcal{K}$ be an algebraically closed field of characteristis 0, completed with respect to a non-Archimedean absolute value (field $\mathbb{C}_p$ is an example), $\mathbb{W}$ is $\mathbb{C}$ or $\mathcal{K}$ and let $\mathbb{P}^n(\mathbb{W})$ be the projective space of dimension n over $\mathbb{W}$.

Definition 1.2. Let f be a holomorphic curve from $\mathbb{W}$ into $\mathbb{P}^n(\mathbb{W})$, then there exist holomorphic functions $f_0, \ldots, f_n$ on $\mathbb{W}$, where there is at least one non-uniform function equal to zero such that

$$f(z) = (f_0(z) : \cdots : f_n(z))$$

for every $z \notin \{f_0 = \cdots = f_n = 0\}$. We call $(f_0, \ldots, f_n)$ be a *representation* of curve f. If functions $f_0, \ldots, f_n$ without common zeros on $\mathbb{W}$ then we call $(f_0, \ldots, f_n)$ be a *reduced representative* of f.

Definition 1.3. The holomorphic curves $f : \mathbb{W} \to \mathbb{P}^n(\mathbb{W})$ is called *linearly degenerate* if there exists some real linear subspace H of $\mathbb{P}^n(\mathbb{W})$ such that $f(\mathbb{W}) \subset H$. The holomorphic curves f is called *algebracally degenerate* if there exists some real algebraic subset G of $\mathbb{P}^n(\mathbb{W})$ such that $f(\mathbb{W}) \subset G$.

Let X be a k-dimensional linear projective subspace of $\mathbb{P}^n(\mathbb{W})$, $(1 \leqslant k \leqslant n)$ and some positive integer N. Let $\{H_1, \ldots, H_q\}$ be a collection of $q \geqslant N+1$ hyperplanes in $\mathbb{P}^n(\mathbb{W})$, where H_j is defined by linear form L_j, $1 \leqslant j \leqslant q$. A collection $\{H_1, \dots, H_q\}$ is said to be *in $N-$subgeneral position with X* if for any subset $\{i_0, \dots, i_N\}$ of $\{1, \dots, q\}$ of cardinality $N+1$, we have

$$\{x \in X : L_{i_j}(x) = 0,\ j = 0, \dots, N\} = \emptyset. \tag{1.1}$$

When $k = n$, the collection $\{H_1, \dots, H_q\}$ is said to be *in $N-$subgeneral position* (with $\mathbb{P}^n(\mathbb{W})$). If $N = n = k$, the collection $\{H_1, \dots, H_q\}$ is said to be *in general position.*

1.2. Cartan's second main theorem

Let $f_0, \dots, f_n$ are entire functions, linear independent without common zeros on $\mathbb{W}$. Let $\mathcal{L} = \mathcal{L}(f_0, \dots, f_n)$ be the set of all of non-trivial linear combinations of $f_0, \dots, f_n$. For z_0 is an arbitrary point on $\mathbb{W}$, in 2014, Anderson and Hinkkanen proved:

Lemma 1.4. For any $z_0 \in \mathbb{C}$, the possible orders of zeros of functons in $\mathcal{L} = \mathcal{L}(f_0, \dots, f_n)$ at z_0 form the sequence $d_0(z_0), d_1(z_0), \dots, d_n(z_0)$ such that

$$0 = d_0(z_0) < d_1(z_0) < \cdots < d_n(z_0).$$

Lemma 1.4 gives us a property about the possible orders of functions belonging to $\mathcal{L} = \mathcal{L}(f_0, \dots, f_n)$ in the case of holomorphic functions on $\mathbb{C}$ at an arbitrary point. Note that Lemma 1.4 is also true for the non-Archimedean case.

Definition 1.9. The $d_0(z_0), d_1(z_0), \dots, d_n(z_0)$, in the above lemma is said to be the *characteristic exponent* of $f_0, \dots, f_n$ at z_0.

The following lemma shows the relationship between the Wronskian of holomorphic functions and their characteristic exponent sequence:

Lemma 1.5. Let $f_0, \dots, f_n$ are entire functions on $\mathbb{C}$ without common zeros, at least one of them is non-constant. For any $z_0 \in \mathbb{C}$, call $d_0(z_0), \dots, d_n(z_n)$ is characteristic exponent of $f_0, \dots, f_n$ at z_0. Then

i) If $W(z_0) \neq 0$ then $d_0(z_0) = 0 < d_1(z_0) < \cdots < d_n(z_0) = n$;

ii) If $W(z_0) = 0$ then $d_0(z_0) = 0 < d_1(z_0) < \cdots < d_n(z_0)$ depend on z_0, furthermore the order of the zeros of W at z_0 is equal to

$$\sum_{j=1}^{n} d_j(z_0) - \frac{n(n+1)}{2}.$$

Lemma 1.5 was proved by Anderson and Hinkkanen in 2014 for the case of complex meromorphic functions. Note that this is still true in the case of non-Archimedean holomorphic functions.

Lemma 1.6. Let $f : \mathbb{W} \to \mathbb{P}^n(\mathbb{W})$ be a linearly non-degenerate holomorphic curve and $f = (f_0 : \cdots : f_n)$ be a reduced representative of f. Let $N \geqslant n$ is an integer. Assuming $\mathcal{H} = \{H_1, \ldots, H_q\},\ q \geqslant N+1$ be a collection of hyperplanes in $N-$subgeneral position in $\mathbb{P}^n(\mathbb{W})$. Then

$$\sum_{j=1}^{q} \varepsilon(H_j, z_0) \leqslant (N - n + 1)\mathrm{ord}_W(z_0).$$

Let $f = (f_0 : \cdots : f_n) : \mathcal{K} \to \mathbb{P}^n(\mathcal{K})$ is a non-Archimedean holomorphic curve and H be a hyperplane in $\mathbb{P}^n(\mathcal{K})$. We denote

$$\nu_f(r, H) = \sum_{|z| \leqslant r} \nu(H, z).$$

Definition 1.10. The integrated reduced counting function of f is defined by

$$\mathcal{N}_f(r, H) = \int_0^r \frac{\nu_f(t, H) - \nu_f(0, H)}{t} dt + \nu_f(0, H) \log r.$$

Easy to see

$$\nu_f(r, H) \leqslant n_f^n(r, H)$$

So we have

$$\mathcal{N}_f(r, H) \leqslant N_f^n(r, H).$$

Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a collection of $H_1, \ldots, H_q$ hyperplanes in general position in $\mathbb{P}^n(\mathcal{K})$ and let L_j is the linear form that defining H_j, $j = 1, 2, \ldots, q$. Set

$$H = \frac{L_1(f)L_2(f)\ldots L_q(f)}{W},$$

where W is the Wronskian determinant of $f_0, f_1, \ldots, f_n$. From Lemma 1.6, for each arbitrary z_0 we always have

$$\sum_{j=1}^{q} \varepsilon(H_j, z_0) \leqslant \mathrm{ord}_W(z_0).$$

Set

$$\mathcal{V}(\mathcal{H}, z) = \mathrm{ord}_W(z_0) - \sum_{j=1}^{q} \varepsilon(H_j, z) \geqslant 0.$$

And denote

$$\mathcal{V}_f(r, \mathcal{H}) = \sum_{|z| \leqslant r} \mathcal{V}(\mathcal{H}, z).$$

Definition 1.11. Call

$$\mathcal{U}_f(r, \mathcal{H}) = \int_0^r \frac{\mathcal{V}_f(t, \mathcal{H}) - \mathcal{V}_f(0, \mathcal{H})}{t} dt - \mathcal{V}_f(0, \mathcal{H}) \log r$$

the *counting function* of the unrealized excesses for $\mathcal{H}$.

The following theorem is a Cartan type form of the second main theorem for non-Archimedean holomorphic curves with reduced counting functions.

Theorem 1.7. *Let $f = (f_0 : \cdots : f_n) : \mathcal{K} \to \mathbb{P}^n(\mathcal{K})$ be a linearly non-degenerate holomorphic curve, and let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a collection of $q \geqslant n+1$ hyperplanes in $\mathbb{P}^n(\mathcal{K})$ in general position. Then we have*

$$\begin{aligned}(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} \mathcal{N}_f(r, H_j) - \mathcal{U}_f(r, \mathcal{H}) - N(r, H) \\ - \frac{n(n+1)}{2} \log r + O(1),\end{aligned}$$

as $r \to \infty$ outside a set of finite linear measure.

Notice 1.2. For each holomorphic curve f and for each hyperplane H. From the definition we can easily see

$$\mathcal{N}_f(r, H) \leqslant N_f^n(r, H).$$

Therefore, our inequality in Theorem 1.7 is stronger than the inequality in Theorem B of H. H. Khoai and M. V. Tu.

Chapter 2
Some forms of the second main theorem for holomorphic curves

2.1. Cartan-Nochka type theorem for curves on the non-Archimedean field

For each finite set R, we denote $|R|$ is the number of elements of R. In 1983, E. I. Nochka have proved the following two results related to collection of hypersurfaces in $N-$subgeneral position.

Lemma 2.1. Let $q > 2N - n + 1$ and $\{H_j\}_{j=1}^{q}$ be a collection of hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in $N-$subgeneral position. Denote $Q = \{1, 2, \ldots, q\}$. Then there are positive rational constants $w(j)$ satisfying the following:

i) $0 < w(j) \leqslant 1$ for any $j \in Q$;

ii) Setting $w^* = \max_{j\in Q} w(j)$, we have

$$\sum_{j=1}^{q} w(j) = w^*(q - 2N + n - 1) + n + 1;$$

iii) $\dfrac{n+1}{2N-n+1} \leqslant w^* \leqslant \dfrac{n}{N};$

iv) For each $R \subset Q$ satisfy $0 < |R| \leqslant N + 1$, we have

$$\sum_{j\in R} w(j) \leqslant \operatorname{rank}\{H_j\}_{j\in R};$$

Rational numbers are not negative $w(j), j = 1, \ldots, q$, in Lemma 2.1 is called *Nochka weights* and w^* is called *Nochka constant.*

Lemma 2.2. Let $q > 2N - n + 1$ and $\{H_j\}_{j=1}^{q}$ be a collection of hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in $N-$subgeneral position. Let $w(j), j \in Q$, are the Nochka weights in Lemma 2.1. Denote $Q = \{1, 2, \ldots, q\}$ and $E_j \geqslant 1, j \in Q$, are arbitrary real numbers. Then, for each $R \subset Q$ with $0 < |R| \leqslant N+1$, there is a subset $R_0 \subset R$ such that

$$|R_0| = \operatorname{rank}\{H_j\}_{j\in R_0} = \operatorname{rank}\{H_j\}_{j\in R},$$

and

$$\prod_{j\in R} E_j^{w(j)} \leqslant \prod_{j\in R_0} E_j.$$

Note that, Lemmas 2.1 and 2.2 still holds true in the case collection of hypersurfaces in $N-$subgeneral position in the space $\mathbb{P}^n(\mathcal{K})$.

Let H be a hyperplane in $\mathbb{P}^n(\mathcal{K})$, we repeat for each $z_0 \in \mathcal{K}$, $\varepsilon(H, z_0)$ is the excess of $H_j \circ f$ at z_0, i.e. $\varepsilon(H, z_0) = d_j(z_0) - j \geqslant 0$ as $\operatorname{ord}_{H_j\circ j}(z_0) = d_j(z_0)$. We have the following lemma on the relationship between the summation of the excesses with Nochka weights of the hyperplanes at z_0 and the order of Wronskian of f at z_0 in the case of non-Archimedean.

Lemma 2.3. Let $f : \mathcal{K} \to \mathbb{P}^n(\mathcal{K})$ be a linearly non-degenerate holomorphic curve, and let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a collection of $q > 2N - n + 1$ hyperplanes in $\mathbb{P}^n(\mathcal{K})$ in $N-$subgeneral position. Let $w(j), j = 1, \ldots, q$, are the Nochka weights in Lemma 2.1. Then for any $z_0 \in \mathbb{K}$, we have

$$\sum_{j=1}^{q} w(j)\varepsilon(H_j, z_0) \leqslant \operatorname{ord}_{W_f}(z_0). \tag{2.1}$$

where $\operatorname{ord}_{W_f}(z_0)$ is order of W_f at z_0.

Let $f : \mathcal{K} \to \mathbb{P}^n(\mathcal{K})$ be a holomorphic curve, where $(f_0, \ldots, f_n)$ be a reduced representative of f. Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a collection $q \geqslant 2N - n + 1$ hyperplanes in $\mathbb{P}^n(\mathcal{K})$ in $N-$subgeneral position. Let $w(j), j = 1, \ldots, q$, are Nochka weights of $\mathcal{H}$ in Lemma 2.1. For any $z \in \mathcal{K}$, we set

$$\mathcal{V}(\mathcal{H}, z) = \operatorname{ord}_{W_f}(z_0) - \sum_{j=1}^{q} w(j)\varepsilon(H_j, z),$$

here $W_f = W(f_0, \ldots, f_n)$. It is easy see that $\mathcal{V}(\mathcal{H}, z) \geqslant 0$ by Lemma 2.3. For $r \geqslant 0$, we set

$$\mathcal{V}_f(r, \mathcal{H}) = \sum_{|z|\leqslant r} \mathcal{V}(\mathcal{H}, z)$$

and call

$$\mathcal{U}_f(r,\mathcal{H})=\int_0^r\frac{\mathcal{V}_f(t,\mathcal{H})-\mathcal{V}_f(0,\mathcal{H})}{t}dt-\mathcal{V}_f(0,\mathcal{H})\log r$$

the *counting function of the unrealized excesses with Nochka weights* for $\mathcal{H}$.

For any $j=1,\dots,q$, assume that

$$w(j)=a_j/b_j$$

here a_j,b_j are non-negative integers and $b_j\neq 0$. Set $M=b_1\dots b_q$. Set

$$\Phi=\frac{L_1(f)^{Mw(1)}\dots L_q(f)^{Mw(q)}}{W_f^M},\tag{2.5}$$

here $W_f=W(f_0,\dots,f_n)$.

In 2023 we proved a form of the second main theorem with a reduced counting function for a non-Archimedean holomorphic curve associated with a collection of hyperplanes in $N-$subgeneral position in projective space $\mathbb{P}^n(\mathcal{K})$ as follows:

Theorem 2.4. *Let $f:\mathcal{K}\to\mathbb{P}^n(\mathcal{K})$ be a linearly non-degenerate holomorphic curve, and let $\mathcal{H}=\{H_1,\dots,H_q\}$ be a collection of $q\geqslant 2N-n+1$ hyperplanes in $\mathbb{P}^n(\mathcal{K})$ in $N-$subgeneral position. Then we have*

$$\begin{aligned}(q-2N+n-1)T_f(r)\leqslant\sum_{j=1}^q\mathcal{N}_f(r,H_j)-\frac{N}{n}\mathcal{U}_f(r,\mathcal{H})-\frac{N}{Mn}N(r,\Phi)\\-\frac{(N+1)n}{2}\log r+O(1),\end{aligned}$$

as $r\to\infty$ outside a set of finite linear measure.

Note that, since $\mathcal{N}_f(r,H_j)\leqslant N_f^n(r,H_j)$ for $j=1,2,\dots,q$, so Theorem 2.4 is an improved version of Cartan-Nochka's theorem (Theorem 1.3) for non-Archimedean holomorphic curves. Furthermore when $\mathcal{H}$ is in general position, then $N=n$ and we can choose $w(j)=1$ for $j=1,\dots,q$, so $M=1$, $\Phi=H$ and the counting function of the unrealized excesses with Nochka weights $\mathcal{U}_f(r,\mathcal{H})$ coincides with the residual counting function. In this case Theorem 2.4 yields Theorem 1.7.

2.2. Theorem for the curve on an annulus

Let $R_0>1$ be a fixed positive real number or $+\infty$, set

$$\Delta=\Big\{z\in\mathbb{C}:\frac{1}{R_0}<|z|<R_0\Big\},$$

be an annulus in $\mathbb{C}$. For each real number r satisfying $1 < r < R_0$, we denote

$$\Delta_{1,r} = \left\{z \in \mathbb{C} : \frac{1}{r} < |z| \leqslant 1\right\},\ \Delta_{2,r} = \left\{z \in \mathbb{C} : 1 < |z| < r\right\},$$
$$\Delta_r = \left\{z \in \mathbb{C} : \frac{1}{r} < |z| < r\right\}.$$

Let D be a hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d and Q be the homogeneous polynomial in $\mathbb{C}[x_0, \ldots, x_n]$ of degree d defining D, then

$$Q(z_0, \ldots, z_n) = \sum_{k=0}^{n_d} a_k z_0^{i_{k0}} \ldots z_n^{i_{kn}},$$

where $n_d = \binom{n+d}{n} - 1$ and $i_{k0} + \cdots + i_{kn} = d$ for $k = 0, \ldots, n_d$, we denote

$$(f, D) = Q(f) = \sum_{k=0}^{n_d} a_k f_0^{i_{k0}} \ldots f_n^{i_{kn}}.$$

Next we prove a form of the basic theorem for holomorphic curves on an annulus for the truncated multiple counting function with the target being the hypersurfaces in general position, necessary for proving the uniqueness theorem. for the holomorphic curve on an annulus in Chapter 3. We first introduce Wronskian, introduce and demonstrate some related knowledge.

Let $f_0, \ldots, f_n$ be holomorphic functions, we denote by $W(f_0, \ldots, f_n)$ is the Wronskian of the $f_0, \ldots, f_n$, namely

$$W(f_0, \ldots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{vmatrix}.$$

Proposition 2.10. Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve and $(f_0 : \cdots : f_n)$ is a reduced representative of f. Then we have for any $1 < r < R_0$

$$\| \quad m_0\left(r, \frac{W(f_0, \ldots, f_n)}{f_0 \cdots f_n}\right) = O_f(r). \tag{2.27}$$

We recall that hypersurfaces $D_1, \ldots, D_q,\ q > n$, in $\mathbb{P}^n(\mathbb{C})$ are said to be in general position if for any distinct $i_1, \ldots, i_{n+1} \in \{1, \ldots, q\}$,

$$\bigcap_{k=1}^{n+1} \operatorname{supp}(D_{i_k}) = \emptyset.$$

Proposition 2.11. Let $D_1, \dots, D_q$ be q hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of the common degree d in general position in $\mathbb{P}^n(\mathbb{C})$. Put $M = \binom{n+d}{n} - 1$. Then there exist $(M-n)$ hypersurfaces $T_1, \dots, T_{M-n}$ in $\mathbb{P}^n(\mathbb{C})$ such that for any subset $R \subset \{1, \dots, q\}$ with $\#R = \operatorname{rank}\{D_j\}_{j\in R} = n+1$, then $\operatorname{rank}\{\{D_j\}_{j\in R} \cup \{T_j\}_{j=1}^{M-n}\} = M+1$.

Note that the proof of Proposition 2.11 shows that the hypersurfaces $T_j, j = 1, \dots, M-n$ have the same degree d.

In 2022, H.T. Phuong and L. Vilaisavanh proved the following theorem called the first main theorem for holomorphic curves on an annulus.

Proposition 2.12. Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and $f = (f_0 : \cdots : f_n) : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image does not contain D. Then we have for any $1 < r < R_0$

$$m_f(r, D) + N_f(r, D) = dT_f(r) + O(1).$$

Now we prove a form of second main theorem for holomorphic curves on an annulus combined with hypersurfaces.

Theorem 2.13. *Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic curve, and let $D_j, 1 \leqslant j \leqslant q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Let d is the least common multiple of the d_j and set $M = \binom{n+d}{n} - 1$. Then, for any $1 < r < R_0$ and $q \geqslant M+1$, we have*

$$\| \quad (q - M - 1)T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_f^M(r, D_j) + O_f(r). \tag{2.32}$$

Chapter 3
Uniqueness theorem for holomorphic curves on an annulus

3.1. Chen-Yan type uniqueness theorem

In this section we prove a uniqueness form of Yan-Chen type theorem for holomorphic curves on an annulus. To prove the uniqueness theorem, we need the following supporting results:

Proposition 3.1. Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic curve and D_1, D_2 be distinct hypersurfaces of the same degree d. Then

$$T_0\left(r, \frac{(f, D_1)}{(f, D_2)}\right) \leqslant dT_f(r) + O(1), \tag{5}$$

for any r such that $1 < r < R_0$.

For $f, g : \Delta \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic curves, we denote

$$T(r) = T_f(r) + T_g(r).$$

For each $D_j \in \mathcal{D}$ hypersurfaces, where $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of hypersurfaces and δ be a positive integer, we set

$$F_j(\delta) = \sum_{t=1}^{\delta-1}(\delta - t)\overline{N}_{f,=t}(r, D_j); \quad G_j(\delta) = \sum_{t=1}^{\delta-1}(\delta - t)\overline{N}_{g,=t}(r, D_j).$$

Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of q hypersurfaces in general position. We denote the degree of D_j by d_j for $j = 1, \ldots, q$ and let d be the least common multiple of the d_j. Set $M = \binom{n+d}{d} - 1$. The following theorem is a unique form of Yan-Chen type theorem for holomorphic curves on an annulus that we announced in 2023.

Theorem 3.2. *Let $\mathcal{D} = \{D_1, \dots, D_q\}$ be a collection of q hypersurfaces in general position and $f, g : \Delta \to \mathbb{P}^n(\mathbb{C})$ be algebraically non-degenerate holomorphic curves such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Assume that*

a) $\overline{E}_f(D_i) \cap \overline{E}_f(D_j) = \emptyset$ *for any* $i \neq j \in \{1, \dots, q\}$*;*

b) $\overline{E}_f(D_j) \subset \overline{E}_g(D_j)$ *for any* $j = 1, 2, \dots, q$ *and* $f(z) = g(z)$ *for all* $z \in \overline{E}_f(\mathcal{D})$.

c) $\liminf\limits_{r \to R_0} \sum_{j=1}^q N_f^1(r, D_j) / \sum_{j=1}^q N_g^1(r, D_j) > \dfrac{M}{M+1}$.

If $q \geqslant 2M + 3$, then there exists a subset $S \subset \{1, \dots, q\}$ such that $\#S > M + 1$ and

$$\frac{(f, D_k)^{d/d_k}}{(f, D_l)^{d/d_l}} \equiv \frac{(g, D_k)^{d/d_k}}{(g, D_l)^{d/d_l}} \quad \text{for all } k \neq l \in S. \tag{3.6}$$

To prove Theorem 3.2, we need the following Proposition.

Proposition 3.3. Under the assumption of Theorem 3.2 and let $D_1, \dots, D_q$ have the same degree d. Then for any positive integer $\delta > 0$ and for any $k \neq l \in \{1, \dots, q\}$ such that $\Phi = \dfrac{(f, D_k)}{(f, D_l)} - \dfrac{(g, D_k)}{(g, D_l)} \not\equiv 0$, we have

$$\begin{aligned} &\delta N_f^1(r, D_k) + \delta N_f^1(r, D_l) + \sum_j N_f^1(r, D_j) \\ &\leqslant dT(r) + F_k(\delta) + F_l(\delta) + G_k(\delta) + G_l(\delta) + O(1), \end{aligned} \tag{3.7}$$

where $1 < r < R_0$ and the sum $\sum_j$ is taken over $j \in \{1, \dots, q\} \backslash \{k, l\}$.

3.2. Fujimoto-style uniqueness theorem

To prove the main result, we first introduce some knowledge about counting functions and a form of second main theorem used for proving theorems. Let f be a holomorphic curve on Δ, D be a hypersurface and Q is the homogeneous polynomial defining D. For any real number $r : 1 < r < R_0$, for any positive integer k and α, we define

$$n_{1,f}^{\alpha}(r, D, \leqslant k) = \sum_{z \in \Delta_{1,r}, 0 < \mathrm{ord}_{Q(f)}(z) \leqslant k} \min\{\mathrm{ord}_{Q(f)}(z), \alpha\},$$
$$n_{2,f}^{\alpha}(r, D \leqslant k) = \sum_{z \in \Delta_{2,r}, 0 < \mathrm{ord}_{Q(f)}(z) \leqslant k} \min\{\mathrm{ord}_{Q(f)}(z), \alpha\}$$

and

$$N_{1,f}^{\alpha}(r, D, \leqslant k) = \int_{r^{-1}}^{1} \frac{n_{1,f}^{\alpha}(t, D, \leqslant k)}{t} dt,$$
$$N_{2,f}^{\alpha}(r, D, \leqslant k) = \int_{1}^{r} \frac{n_{2,f}^{\alpha}(t, D, \leqslant k)}{t} dt.$$

We set

$$N_{f,\leqslant k}^{\alpha}(r, D) = N_f^{\alpha}(r, D, \leqslant k) := N_{1,f}^{\alpha}(r, D, \leqslant k) + N_{2,f}^{\alpha}(r, D, \leqslant k).$$

Similarly, we denote

$$n_{1,f}^{\alpha}(r, D, \geqslant k) = \sum_{z \in \Delta_{1,r}, \mathrm{ord}_{Q(f)}(z) \geqslant k} \min\{\mathrm{ord}_{Q(f)}(z), \alpha\},$$
$$n_{2,f}^{\alpha}(r, D \geqslant k) = \sum_{z \in \Delta_{2,r}, \mathrm{ord}_{Q(f)}(z) \geqslant k} \min\{\mathrm{ord}_{Q(f)}(z), \alpha\}.$$

We define

$$N_{1,f}^{\alpha}(r, D, \geqslant k) = \int_{r^{-1}}^{1} \frac{n_{1,f}^{\alpha}(t, D, \geqslant k)}{t} dt,$$
$$N_{2,f}^{\alpha}(r, D, \geqslant k) = \int_{1}^{r} \frac{n_{2,f}^{\alpha}(t, D, \geqslant k)}{t} dt.$$

Set

$$N_{f,\geqslant k}^{\alpha}(r, D) = N_f^{\alpha}(r, D, \geqslant k) := N_{1,f}^{\alpha}(r, D, \geqslant k) + N_{2,f}^{\alpha}(r, D, \geqslant k).$$

So it is to see that

$$N_f^{\alpha}(r, D) = N_f^{\alpha}(r, D, \leqslant k) + N_f^{\alpha}(r, D, \geqslant k+1)$$

holds for any integer α and k.

The following statement is a form of second main theorem for holomorphic curves proposed by H.T. Phuong and L. Vilaisavanh proof in 2022, necessary for proving the uniqueness theorem forms in this section.

Proposition 3.4. Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic curve, and let $D_j, 1 \leqslant j \leqslant q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Let d be the least common multiple of the $d_1, d_2, \dots, d_q$. For $0 < \varepsilon < 1$ and

$$\alpha \geqslant \left(d[(n+1)^2 2^n)\varepsilon^{-1}] + 1\right)^n.$$

Then for any $1 < r < R$, we have

$$\| \quad (q - (n+1) - \varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} d_j^{-1} N_f^{\alpha}(r, D_j) + O_f(r).$$

Let $\mathcal{D} = \{D_1, \dots, D_q\}$ be a collection of q hypersurfaces in general position. We denote the degree of D_j by d_j for $j = 1, \dots, q$ and let d be the least common multiple of the d_j. Set $\delta_{\mathcal{D}} := \min\{d_1, \dots, d_q\}$ and $M = (d(n+1)^2 2^{n+1} + 1)^n$. The following two theorems are the uniqueness theorem forms for holomorphic curves on Fujimoto-type annulus.

Theorem 3.5. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \dots, D_q\}$ be a collection of $q > n + 1 + 2Mn/\delta_{\mathcal{D}}$ hypersurfaces in general position $\mathbb{P}^n(\mathbb{C})$ such that $f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$. Then $f \equiv g$.*

Theorem 3.6. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \dots, D_q\}$ be a collection of $q > n + 1 + 2M/\delta_{\mathcal{D}}$ hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$ such that*

(a) *$f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$,*

(b) *$\overline{E}_f(D_i) \cap \overline{E}_f(D_j) = \emptyset$ and $\overline{E}_g(D_i) \cap \overline{E}_g(D_j) = \emptyset$ for all $i \neq j \in \{1, \dots, q\}$.*

Then $f \equiv g$.

CONCLUSION

In this thesis, we have studied some forms of the Fundamental Theorem with reduced counting functions or truncated multiple counting functions for holomorphic curves on images in the field $\mathbb{W}$ and the uniqueness problem for holomorphic curves on an annulus in the complex plane $\mathbb{C}$.

The main results of the thesis include:

1. Proving two forms of the Second Main Theorem for holomorphic curves on the non-Archimedean field with the reduced counting function in two cases where the targets are hyperplanes in general position (Theorem 1.7) and in subgeneral position (Theorem 2.4).

2. Construction of a form of the Second Main Theorem for holomorphic curves on the complex field $\mathbb{C}$ in the case of holomorphic curves on an algebraically non-degenerate annulus with a truncated multiple count function combined with hypersurfaces in general position (Theorem 2.13).

3. Give three new theorems on the uniqueness problem of holomorphic curves on an annulus with the target being hypersurfaces in general positions (theorems 3.2, 3.5, 3.6).

We propose some future research directions:

1. Study some Second Main Theorems for holomorphic curves with reduction in different cases of the target.

2. Use the results of the forms of the Second Main Theorem with reduction to study a problem uniqueness to holomorphic curves.

LIST OF THE SCIENTIFIC PUBLICATIONS RELATED OF THE THESIS

[1] Phuong H. T., Hung B. T., Padaphet I. (2023), A version of Cartan-Nochka's theorem for non-Archimedean holomorphic curves with integrated reduced counting functions, *Submitted*.

[2] Phuong H. T., Ninh, L. Q., Padaphet I., (2019), On the Nevanlinna-Cartan Second Main Theorem for non-Archimedean Holomorphic Curves, *p-Adic Numbers, Ultrametric Analysis and Applications*, Vol. 11, No. 4, pp. 299 - 306.

[3] Phuong H. T., Padaphet I. (2023), A uniqueness theorem for holomorphic curves on annulus sharing hypersurface, *Accepted to Complex Variables and Elliptic Equations,* Doi: 10.1080/17476933.2023.2234830.

[4] Phuong H. T., Padaphet I., Ninh, L. Q., (2022), Some uniqueness theorems for holomorphic curves on annulus sharing hypersurfaces in general position, *East-West J. of Mathematics*, Vol. 23, No 2, pp. 100-111.